

Solutions of Navier-Stokes Equations with Coriolis Force in the Rotational Framework

Nghiệm của các phương trình Navier-Stokes với lực Coriolis trong môi trường khung quay

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Abstract

In this article, for $0 \leq m < \infty$ and the index vectors $q = (q_1, q_2, q_3), r = (r_1, r_2, r_3)$ where $1 \leq q_i \leq \infty, 1 < r_i < \infty$ and $1 \leq i \leq 3$, we study new results of Navier-Stokes equations with Coriolis force in the rotational framework in mixed-norm Sobolev-Lorentz spaces $\dot{H}^{m,r,q}(R^3)$, which are more general than the classical Sobolev spaces. We prove the existence and uniqueness of solutions to the Navier-Stokes equations (NSE) under Coriolis force in the spaces $L^\infty([0, T]; \dot{H}^{m,r,q})$ by using topological arguments, the fixed point argument and interpolation inequalities. We have achieved new results compared to previous research in the Navier-Stokes problems.

Keywords: Navier-stokes equations, coriolis force, rotation framework.

Tóm tắt

Trong bài báo này, cho $0 \leq m < \infty$ và các vectơ chỉ số $q = (q_1, q_2, q_3), r = (r_1, r_2, r_3)$, với $1 \leq q_i \leq \infty, 1 < r_i < \infty$ và $1 \leq i \leq 3$, chúng tôi nghiên cứu những kết quả mới của các phương trình Navier-Stokes với lực ngoài Coriolis trong không gian chuẩn kết hợp Sobolev-Lorentz $\dot{H}^{m,r,q}(R^3)$, không gian tổng quát hơn không gian Sobolev cổ điển đã biết. Chúng tôi chứng minh sự tồn tại và tính duy nhất nghiệm của các phương trình Navier-Stokes (NSE) dưới tác động của lực Coriolis trong không gian $L^\infty([0, T]; \dot{H}^{m,r,q})$ bằng cách sử dụng tính chất các không gian, lý thuyết điểm bất động và các bất đẳng thức nội suy. Chúng tôi đã đạt được những kết quả mới so với các nghiên cứu trước đó về các bài toán Navier-Stokes.

Từ khóa: Các phương trình Navier-stokes, lực Coriolis, khung quay.

1. Introduction

The Coriolis force arises in almost all of the models of meteorology and geophysics dealing with large-scale phenomena. If we look around to see what is happening, we realize that the rotation leads to many interesting phenomena, this effect is substantial: the climate is affected and complicated because the rotation changes the flow of air heated by the sun. Typhoons and their motion are another phenomenon resulting from the effect of rotation of Earth. The flows of the ocean can be affected by the rotation of Earth as well. Changing the motion of moving objects of fluids such as air and ocean illustrated above is caused by force. This force on all moving bodies is called Coriolis force. Almost all of the fluids following on Earth can be affected, so researching the Coriolis effects in Navier-Stokes equations are very important. We consider Navier-Stokes equations in the rational

framework with Coriolis force in R^3 . For the recent results concerning periodic solutions for the Navier-Stokes equations in the rotational framework, we refer to the work of Kozono, Mashiko and Takada [1]. They proved the existence of unique mild periodic solution with suitable smallness assumptions on forces.

The problems of mild solutions to the Navier-Stokes equations and the problems concerning Navier-Stokes equations have a long history, first results on constructing mild solutions for the Navier-Stokes equations in the Sobolev spaces $H^s(R^3)$, go back to Kato and Fujita [2].

In 1995, using the method of the bilinear operator, Cannone [3] proved the existence of mild solutions in $\dot{H}^m(R^3)$, Giga [4] continued to establish

the mild solutions with value in $L^q(R^3)$ and Cannone and Meyer [3,5] studied the existence of mild solutions with initial data in the space L^∞ . Base on the Besov spaces on a bounded domain in R^3 , the regularity and uniqueness conditions for weak solutions of NSE were established in [6]. Kato and Ponce [7] showed that the Navier-Stokes problems are well-posed when the initial data in the homogeneous Sobolev spaces.

Recently, D.Q. Khai and N.M. Tri considered the mild solutions of Navier-Stokes in mixed-norm Sobolev-Lorentz spaces and Sobolev-Fourier-Lorentz spaces, as in [8,9]

This paper approaches to the methods to solve the problems of existence and uniqueness of the solution of the Navier-Stokes equations with Coriolis forces in new spaces $L^\infty([0, T]; \dot{H}^{m,r,q})$. It develops the results in [8] which shows the existence and uniqueness of the solutions of Navier-Stokes equations in the same spaces.

We consider the Navier-Stokes equations in the rotational framework which are given by the equation

$$\begin{cases} u_t + (u \cdot \nabla)u - \Delta u + \nabla p = \omega e_3 \times u + \operatorname{div} F \\ \nabla \cdot u = 0 \\ u(0) = u_0 \end{cases} \quad (1)$$

where ω is the speech of rotation and e_3 -the unit vector in x_3 -direction. The unknown quantities are the velocity $u(t, x) = (u_1(t, x), u_2(t, x), u_3(t, x))$ of the fluid element at time t and position x with the pressure $p(t, x)$.

Apply the Helmholtz projection P we obtain

$$\begin{cases} u_t - P\Delta u = P(\omega e_3 \times u) + P\operatorname{div}(-uu) + g \\ \nabla \cdot u = 0 \\ u(0) = u_0, g = \operatorname{div} F \end{cases} \quad (2)$$

We first define the definition of mild solution to (2). We mean the function $u(t)$ satisfying

$$u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-\tau)\Delta} P[\nabla \cdot (-uu) + g + \omega e_3 \times u(\tau)] d\tau$$

We note the mixed-norm Sobolev –Lorentz spaces $\dot{H}^{m,r,q}$ is considering as the space $(\sqrt{-\Delta})^{-m} L^{r,q}$, equipped with the norm

$$\|u\|_{\dot{H}^{m,r,q}} = \left\| (\sqrt{-\Delta})^m u \right\|_{L^{r,q}}$$

2. Main results

We recall the following results

Lemma 2.1. Let $q = (q_1, q_2, q_3), r = (r_1, r_2, r_3), m \geq 0$ and $0 < T < \infty$ be such that

$$\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} - 1 < m < \frac{\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3}}{2}, 1 \leq q_i \leq \infty$$

$$2 < \frac{q_i}{1 - \frac{m}{\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3}}} < \infty, 1 \leq i \leq 3$$

Then the bilinear operator

$$B(u, v)(t) = \int_0^t e^{(t-\tau)\Delta} P[\nabla \cdot (-uv)] d\tau$$

is continuous from $L^\infty([0, T]; \dot{H}^{m,r,q}) \times L^\infty([0, T]; \dot{H}^{m,r,q})$ to $L^\infty([0, T]; \dot{H}^{m,r,q})$ and we have the the enequality

$$\|B(u, v)(t)\|_{L^\infty([0, T]; \dot{H}^{m,r,q})} \leq T^{\frac{m+1 - (\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3})}{2}} \|u\|_{L^\infty([0, T]; \dot{H}^{m,r,q})} \times \|v\|_{L^\infty([0, T]; \dot{H}^{m,r,q})}$$

Proof. This lemma is obtained from lemma 3 in [8].

Lemma 2.2. If $u \in \dot{H}^{m,r,q}$ then

$$\|e^{t\Delta} u\|_{L^\infty([0, T]; \dot{H}^{m,r,q})} \leq \|u\|_{L^\infty([0, T]; \dot{H}^{m,r,q})}$$

Proof. See lemma 4 in [8].

Lemma 2.3. Let P is the Helmholtz projection onto the divergence-free fields then P is continuous from $L^{r,q}$ to $L^{r,q}, 1 < r < \infty, 1 \leq q \leq \infty$.

Proof. The operator P in [8] is defined by the Riesz transforms

$$(Pf)_j = f_j + \sum_{1 \leq k \leq 3} R_j R_k f_k$$

and R_j is bounded from L^p to $L^p, 1 < p < \infty$ see in ([10], theorem 6.8 and example, p.54) then P is also bounded from L^p to $L^p, 1 < p < \infty$. In addition, we have the interpolation relation

$$(L^{p_1}, L^{p_2})_{\theta, q} = L^{r, q}, 1 < p_1 < r < p_2, 1 \leq q \leq \infty$$

Using the interpolation theory, we obtain that the operator P is bounded from $L^{r,q}$ to $L^{r,q}$, $1 < r < \infty, 1 \leq q \leq \infty$.

Therefore, based on linearity property, P is continuous from $L^{r,q}$ to $L^{r,q}$ and the proof is completed.

Lemma 2.4. If $u \in \dot{H}^{m,r,q}$ then $\forall 0 < t \leq T$

$$\left\| \int_0^t e^{(t-\tau)\Delta} Pu(\tau) d\tau \right\|_{L^\infty([0, T]; \dot{H}^{m,r,q})} \leq T \|u\|_{L^\infty([0, T]; \dot{H}^{m,r,q})}.$$

Proof. We have

$$\begin{aligned} & \left\| \int_0^t e^{(t-\tau)\Delta} Pu(\tau) d\tau \right\|_{\dot{H}^{m,r,q}} \\ &= \left\| \int_0^t e^{(t-\tau)\Delta} (\sqrt{-\Delta})^m Pu(\tau) d\tau \right\|_{L^{r,q}} \\ &= \left\| \int_0^t \frac{1}{[4\pi(t-\tau)]^{\frac{3}{2}}} \int_{\mathbb{R}^3} e^{\frac{-|\theta|^2}{4(t-\tau)}} (\sqrt{-\Delta})^m Pu(\cdot - \theta) d\theta d\tau \right\|_{L^{r,q}} \\ &\leq \int_0^t \frac{1}{[4\pi(t-\tau)]^{\frac{3}{2}}} \int_{\mathbb{R}^3} e^{\frac{-|\theta|^2}{4(t-\tau)}} \|Pu(\tau, \cdot - \theta)\|_{\dot{H}^{m,r,q}} d\theta d\tau \\ &\leq \|Pu\|_{L^\infty([0, T]; \dot{H}^{m,r,q})} \int_0^t \frac{1}{[4\pi(t-\tau)]^{\frac{3}{2}}} \int_{\mathbb{R}^3} e^{\frac{-|\theta|^2}{4(t-\tau)}} d\theta d\tau \\ &\leq T \|u\|_{L^\infty([0, T]; \dot{H}^{m,r,q})}. \end{aligned}$$

Therefore, we obtain the lemma.

Theorem 2.1.

Let

$$q = (q_1, q_2, q_3), r = (r_1, r_2, r_3), m \geq 0, \|g\|_{L^\infty([0, T]; \dot{H}^{m,r,q})},$$

$$\|u_0\|_{\dot{H}^{m,r,q}}$$

be small enough and $0 < T < \infty$ be such that

$$\begin{aligned} \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} - 1 < m < \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3}, 1 < r_i < \infty \\ 2 < \frac{q_i}{m} < \infty, 1 \leq q_i \leq \infty, 1 \leq i \leq 3 \\ 1 - \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} \end{aligned}$$

Then the equation (2) has a unique mild solution u in a small ball of $L^\infty([0, T]; \dot{H}^{m,r,q})$.

Proof. Consider the ball

$$B_\varepsilon = \left\{ v \in L^\infty([0, T]; \dot{H}^{m,r,q}) : \|v\|_{L^\infty([0, T]; \dot{H}^{m,r,q})} \leq \varepsilon \right\}$$

Let φ be the transformation given as $\varphi(v) = u$

where u is the unique mild solution to equation

$$u_t - P\Delta u = P(\omega e_3 \times v) + P\text{div}(-vv) + g \quad (\text{see theorem 6 in [8]})$$

$$u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-\tau)\Delta} P[\nabla \cdot (-vv) + g + \omega e_3 \times v(\tau)] d\tau$$

We will prove that if ε and $\|g\|_{L^\infty([0, T]; \dot{H}^{m,r,q})}$ are small enough then the transformation φ acts from B_ε is a contraction. To do this, we estimate

$$\begin{aligned} \|u(t)\|_{L^\infty([0, T]; \dot{H}^{m,r,q})} &\leq \|e^{t\Delta} v_0\|_{L^\infty([0, T]; \dot{H}^{m,r,q})} \\ &+ \|B(v, v)\|_{L^\infty([0, T]; \dot{H}^{m,r,q})} \\ &+ \left\| \int_0^t e^{(t-\tau)\Delta} P[g + \omega e_3 \times v(\tau)] d\tau \right\|_{L^\infty([0, T]; \dot{H}^{m,r,q})} \\ &\leq \|v_0\|_{\dot{H}^{m,r,q}} + T^{\frac{m+1 - (\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3})}{2}} \|v\|_{L^\infty([0, T]; \dot{H}^{m,r,q})} \\ &\times \|v\|_{L^\infty([0, T]; \dot{H}^{m,r,q})} + T \|g + \omega e_3 \times v\|_{L^\infty([0, T]; \cdot)} \\ &\leq \varepsilon / 2 + T^{\frac{m+1 - (\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3})}{2}} \varepsilon^2 + T \left(\|g\|_{L^\infty([0, T]; \dot{H}^{m,r,q})} + \varepsilon \right) \\ &\leq \varepsilon \end{aligned}$$

if $\|g\|_{L^\infty([0, T]; \dot{H}^{m,r,q})}, \varepsilon$ are small enough. It implies the transformation φ acts from B_ε into itself.

Next, for v, v^* in B_ε , we have

$$-vv + v^*v^* = (v^* - v)v + v^*(v^* - v)$$

$$\begin{aligned} \varphi(v) - \varphi(v^*) &= \int_0^t e^{(t-\tau)\Delta} P\nabla \cdot \left\{ [(v^* - v)v + v^*(v^* - v)] \right. \\ &\quad \left. + \omega e_3 \times (v - v^*)(\tau) \right\} d\tau \end{aligned}$$

Therefore,

$$\|\varphi(v) - \varphi(v^*)\|_{L^\infty([0, T]; \dot{H}^{m,r,q})}$$

$$\begin{aligned} &\leq \left\| \int_0^t e^{(t-\tau)\Delta} P \nabla \cdot \left\{ \left[(v^* - v)v + v^*(v^* - v) \right] \right. \right. \\ &\quad \left. \left. + \omega e_3 \times (v - v^*)(\tau) \right\} d\tau \right\|_{L^\infty([0, T]; \dot{H}^{m,r,q})} \\ &\leq T^{\frac{m+1 - \left(\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3}\right)}{2}} \left[\|v\|_{L^\infty([0, T]; \dot{H}^{m,r,q})} + \|v^*\|_{L^\infty([0, T]; \dot{H}^{m,r,q})} \right] \\ &\quad \times \|v - v^*\|_{L^\infty([0, T]; \dot{H}^{m,r,q})} \\ &\leq 2\varepsilon T^{\frac{m+1 - \left(\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3}\right)}{2}} \|v - v^*\|_{L^\infty([0, T]; \dot{H}^{m,r,q})} \\ &< \|v - v^*\|_{L^\infty([0, T]; \dot{H}^{m,r,q})} \text{ if } T \text{ is small enough.} \end{aligned}$$

It implies that that φ is a contraction and the equation (2) has unique solution in this ball.

Therefore, the equation (1) has the unique solution in the ball of $L^\infty([0, T]; \dot{H}^{m,r,q})$ under some conditions and we obtain the main results of this paper. In addition, we also continue to study this solution of the global problems in the same space.

3. Conclusion

Our obtained results are considered as an impetus to solve the problems of Navier-Stokes equations in the new Sobolev space which has been presented in [8].

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