

# Near Field and Far Field Calculation from Metallic Elliptical Cylinder Coated with Left-Handed Metamaterial

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## Abstract

Recently, there is an increasing demand for metamaterial research both in theory and practical designs. Metamaterial cloaks and partially filled waveguide have been considered for their potential radiation enhancement and electromagnetic field confinement of sources. For some particular cases, the analysis can be carried out by separation of variables with the use of special functions. This paper present a two-dimensional problem of electromagnetic scattering from line source located outside of a metallic elliptical cylinder covered by isorefractive (right-handed material) and anti-isorefractive dielectric (left-handed material). Analytical solutions of electric and magnetic fields as functions of line source position and layer thickness are discussed in frequency domain.

Keywords: Elliptical cylinder, metamaterial, separation of variables

## 1. Introduction

In recent years, research of left-handed material has been remarkably attention thanks to the fact that dielectric properties of those medias having both negative permittivity and negative permeability. Such characteristics can be manipulated to modify the field distribution inside dielectric medias as well as field scattered from those bodies of evolution [1], [2]. In [3] geometry with sources located inside the materials and there is no presence of metallic core. The exact radiation from electric and magnetic line sources located outside confocal elliptical cylinders with metallic one in the core is investigated in this paper both in near field and far field regions.

The problem of radiation of line source located outside of confocal elliptical cylinders is amenable to an exact solution if linear, homogeneous and isotropic material in each layer has a propagation constant of the infinite medium surrounding the structure [4], [5]. A detailed discussion of these conditions is found in [6], [7]. The purpose of this this work is to analyze the effects of anti-isorefractive to the surrounding space, has on the field trapped inside the layer and on far-fields into infinite series of Mathieu's functions and determining expansion coefficients by imposing boundary conditions at interfaces and on far-field condition. All the solutions are derived in the phasor

domain with a time-dependence factor  $\exp(-i\omega t)$  omitted throughout.

Figure 1 describes the geometry of 2D scattering problem. A metallic elliptical cylinder is coated with a confocal layer made of either isorefractive material (DPS) or anti-isorefractive material (DNG) The Elliptical Cylinder coordinate can be described as follow:

$$x = \frac{d}{2} \cosh(u) \cos(v),$$

$$y = \frac{d}{2} \sinh(u) \cos(v),$$

$$z = z.$$

where  $0 \leq u < \infty, 0 \leq v \leq 2\pi$  and  $-\infty < z < \infty$ .

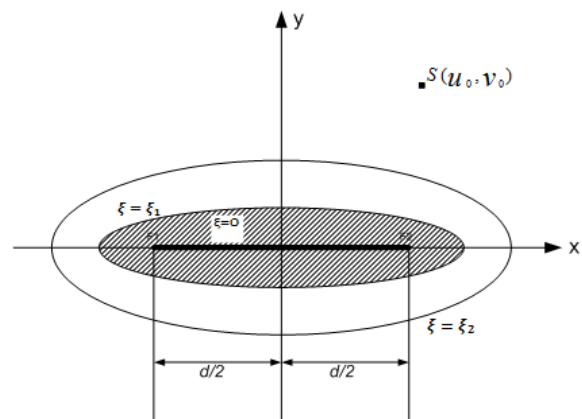


Fig. 1. Geometry of the problems

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This system can be interpreted by  $\xi$  and  $\eta$  where  $\xi = \cosh(u)$  and  $\eta = \cos(v)$ .

When being coated by isorefractive material, the electric permittivity is  $\epsilon_1$  and the magnetic permeability is  $\mu$  whereas for DNG material those are  $-\epsilon_1$  and  $-\mu_1$ . When the material of coating layer is DNG, characteristic impedance  $Z_1$  is always positive but wavenumber, refractive index are always negative [1],[3]. The dimensionless parameter of freespace is  $c = \frac{kd}{2}$ , and  $-c$  in DNG material. To satisfy this eccentricity, permittivity and permeability must follow the condition  $\epsilon_0\mu_0 = \epsilon_1\mu_1$  and the ratio between two intrinsic impedances is indicated as:

$$\zeta_1 = \frac{z_0}{z_1}.$$

The inner and outer surfaces of metallic core and coating layer are indicated as  $u = u_1$  and  $u = u_2$  respectively. The position of line source is illustrated by  $u_0$  and  $v_0$  where  $u_1 < u_2 < u_0$  and  $0 \leq v_0 \leq \frac{\pi}{2}$ .

## 2. Analytical solutions

### 2.1 The case of Electric line source

The electric field of electric line source can be expressed as:

$$E^i = \hat{z} E_{1z}^i = \hat{z} H_0^{(2)}(kR) \quad (1)$$

Where  $H_0^{(2)}$  is the Hankel function of the second kind and  $R$  is the distance of the observation point from the line source. The incident field can be expressed as the function of  $u_0$  and  $v_0$ :

$$E_z^i = 4 \sum_{n=0}^{\infty} \left[ \frac{Re_n^{(1)}(c, u_0) Re_n^{(4)}(c, u_0) Se_n(c, v_0) Se_n(c, v)}{N_n^{(e)}} + \frac{Ro_n^{(1)}(c, u_0) Ro_n^{(4)}(c, u_0) So_n(c, v_0) So_n(c, v)}{N_n^{(o)}} \right] \quad (2)$$

Since the coating layer is either made of isorefractive (DPS) dielectric or anti-isorefractive dielectric (DNG). The Electric field inside the layer can be written as follows:

$$E_{1,z}^{(\pm)} = 4 \sum_{n=0}^{\infty} \left[ \frac{Re_n^{(1)}(c, u_0)}{N_n^{(e)}} (a^{(e),(\pm)} Re_n^{(1)}(\pm c, u) + b^{(e),(\pm)} Re_n^{(4)}(\pm c, u)) Se_n(c, v_0) Se_n(c, v) + \frac{Ro_n^{(1)}(c, u_0)}{N_n^{(o)}} (a^{(o),(\pm)} Ro_n^{(1)}(\pm c, u) + b^{(o),(\pm)} Ro_n^{(4)}(\pm c, u)) So_n(c, v_0) So_n(c, v) \right] \quad (3)$$

The subscript 1 is designated for coating layer, the upper sign (+) stands for the case of DPS while the lower one (-) stands for the case of DNG. The scattered far field can be expressed as:

$$E_z^s = 4 \sum_{n=0}^{\infty} \left[ \frac{c_n^{(e,m)}}{N_n^{(e)}} Re_n^{(1)}(c, u_0) Re_n^{(4)}(c, u) Se_n(c, v_0) \times Se_n(c, v) + \frac{c_n^{(e,m)}}{N_n^{(o)}} Ro_n^{(1)}(c, u_0) Ro_n^{(4)}(c, u) \times So_n(c, v_0) So_n(c, v) \right] \quad (4)$$

Note that:  $\xi = \cosh u$ , and component  $H_v$  can be given by Maxwell equation in Elliptical Coordinate:

$$H_v = \frac{\mp j}{cZ\sqrt{\xi^2 - \eta^2}} \frac{\partial E_z}{\partial u} \quad (5)$$

The upper sign (-) stands for the magnetic field in DPS layer while the lower sign is applied for DNG layer. Such that, we can derive the asymptotic expression of the incident magnetic field:

$$H_v^i = -\frac{-4j}{cZ_0\sqrt{\xi^2 - \eta^2}} \sum_{n=0}^{\infty} \left[ \frac{Re_n^{(1)'}(c, u_0) Re_n^{(4)'}(c, u_0) Se_n(c, v_0)}{N_n^{(e)}} \times Se_n(c, v) + \frac{Ro_n^{(1)'}(c, u_0) Ro_n^{(4)'}(c, u_0) So_n(c, v_0)}{N_n^{(o)}} \times So_n(c, v) \right] \quad (6)$$

Magnetic field inside the layer ( $u_1 < u < u_2$ )

$$H_v^{1,(\pm)} = \frac{\mp 4j}{cZ_1\sqrt{\xi^2 - \eta^2}} \sum_{n=0}^{\infty} \left[ \frac{Re_n^{(1)}(c, u_0)}{N_n^{(e)}} (a^{(e),(\pm)} Re_n^{(1)}(\pm c, u) + b^{(e),(\pm)} Re_n^{(4)}(\pm c, u)) \times Se_n(c, v_0) Se_n(c, v) + \frac{So_n^{(1)}(c, u_0)}{N_n^{(o)}} (a^{(o),(\pm)} Ro_n^{(1)}(\pm c, u) + b^{(o),(\pm)} Ro_n^{(4)}(\pm c, u)) \times So_n(c, v_0) So_n(c, v) \right] \quad (7)$$

The scattered magnetic field can be expressed as:

$$H_v^{s,m} = \frac{-4j}{cZ_0\sqrt{\xi^2 - \eta^2}} \sum_{n=0}^{\infty} \left[ \frac{c_n^{(e,m)}}{N_n^{(e)}} Re_n^{(1)'}(c, u_0) \times Re_n^{(4)'}(c, u) Se_n(c, v_0) Se_n(c, v) + \frac{c_n^{(o,m)}}{N_n^{(o)}} Ro_n^{(1)'}(c, v_0) Ro_n^{(4)'}(c, v) So_n(c, v_0) So_n(c, v) \right] \quad (8)$$

Far field condition can be applied  $\xi \rightarrow \infty$

$$Re_{o,n}^{(4)}(c, \xi) \approx \frac{j^n}{\sqrt{c\xi}} e^{-jc\xi + j\frac{\pi}{4}} \approx \frac{j^n}{\sqrt{kp}} e^{-jc\xi + j\frac{\pi}{4}} \quad (9)$$

where  $p = \sqrt{x^2 + y^2}$ ,  $p|_{\xi \rightarrow \infty} \approx \frac{d}{2}\xi$ , where  $\xi = \cosh(u)$ .

Then, the Electric Scattered Far Field can be written as:

$$E_z^{s,m}|_{\xi \rightarrow \infty} \approx \frac{e^{-jkp}}{\sqrt{kp}} e^{j\frac{\pi}{4}} 4 \sum_{n=0}^{\infty} j^n \left[ \frac{c_n^{(e,m)}}{N_n^{(e)}} Re_n^{(1)}(c, u_0) \times Se_n(c, v_0) Se_n(c, \cos\varphi) + \frac{c_n^{(o,m)}}{N_n^{(o)}} Ro_n^{(1)}(c, u_0) \times So_n(c, v_0) So_n(c, \cos\varphi) \right] \quad (10)$$

The solution for even mode is provided, and that for odd mode is obtained by replacing  $Re_n^{(1),(4)}$  and their

derivatives with  $Ro_n^{(1),(4)}$  and their derivatives. Then the expansion coefficients are retrieved by solving the boundary conditions at  $u = u_1; u = u_2$  for electric field and magnetic field can be written as:

$$E_{1,z}^{(-)}|_{\xi=\xi_1} = 0$$

$$E_{1,z}^{(-)}|_{\xi=\xi_2} = (E_z^{s,2} + E_z^i)|_{\xi=\xi_2},$$

$$H_{1,z}^{(-)}|_{\xi=\xi_2} = (H_v^{s,2} + H_v^i)|_{\xi=\xi_2}.$$

Solving these three equations, the expansion coefficients can be retrieved:

$$a^{(e),(\pm)} = \frac{Re^{(4)}(\pm c, u_1)\alpha}{\Delta(\pm)}, \quad (11)$$

$$b^{(e),(\pm)} = \frac{-Re^{(1)}(\pm c, u_1)\alpha}{\Delta(\pm)}, \quad (12)$$

$$c^{(e),(\pm)} = -\frac{1}{\Delta(\pm)} Re^{(1)}(c, u_0) Re^{(1)'}(c, u_2) \times \\ \times Re^{(4)'}(c, u_0) \Delta_1(\pm) \mp \zeta_1 Re^{(1)}(c, u_2) \times \\ \times Re^{(1)'}(c, u_0) Re^{(4)}(c, u_0) \Delta_2(\pm) \quad (13)$$

And then the notation  $\Delta_1(\pm); \Delta_2(\pm); \alpha$  and  $\Delta(\pm)$  can be expressed as :

$$\Delta_1(\pm) = Re^{(1)}(\pm c, u_1) Re^{(4)}(\pm c, u_2) - \\ - Re^{(1)}(\pm c, u_2) Re^{(4)}(\pm c, u_1) \quad (14)$$

$$\Delta_2(\pm) = Re^{(1)}(\pm c, u_1) Re^{(4)'}(\pm c, u_2) - \\ - Re^{(1)'}(\pm c, u_2) Re^{(4)}(\pm c, u_1) \quad (15)$$

$$\alpha = Re^{(1)}(c, u_0) Re^{(1)'}(c, u_2) Re^{(4)}(c, u_2) \times \\ \times Re^{(4)'}(c, u_0) - Re^{(1)}(c, u_2) Re^{(1)'}(c, u_0) \times \\ \times Re^{(4)}(c, u_0) Re^{(4)'}(c, u_2) \quad (16)$$

The  $\Delta$  is retrieved as:

$$\Delta(\pm) = Re^{(1)}(c, u_0) Re^{(1)'}(c, u_0) [Re^{(4)'}(c, u_2) \times \\ \times \Delta_1(\pm) \mp \zeta_1 Re^{(4)}(c, u_2) \Delta_2(\pm)]. \quad (17)$$

## 2.2 Magnetic line source

Incident magnetic field of a magnetic line source can be expressed as:

$$H^i = \hat{z} H_z^i = \hat{z} H_0^{(2)}(\text{kR}). \quad (18)$$

This incident field can be expressed as in equation [18] electric field of electric line source. The same can be applied to retrieve the scattered magnetic field and approximation of magnetic field with the far field condition. Note that, electric field  $E_v$  is derived from magnetic field by Maxwell's equation in Elliptical Coordinate:

$$E_v = \frac{\pm jZ}{c\sqrt{\xi^2 - \eta^2}} \frac{\partial H_z}{\partial u}, \quad (19)$$

Where  $\xi = \cosh u$ . Such that, the asymptotic expression of the incident electric field:

$$E_v^i = \frac{4jZ_0}{c\sqrt{\xi^2 - \eta^2}} \sum_{n=0}^{\infty} \left[ \frac{Re_n^{(1)'}(c, u_{<}) Re_n^{(4)'}(c, u_{>}) Se_n(c, v_0)}{N_n^{(e)}} \times \right. \\ \left. \times Se_n(c, v) + \frac{Ro_n^{(1)'}(c, u_{<}) Ro_n^{(4)'}(c, u_{>}) So_n(c, v_0) So_n(c, v)}{N_n^{(o)}} \right]. \quad (20)$$

Electric field inside the layer ( $u_1 < u < u_2$ )

$$E_{1,v}^{(\pm)} = \frac{4jZ_0}{c\sqrt{\xi^2 - \eta^2}} \sum_{n=0}^{\infty} \left[ \frac{Re_n^{(1)'}(c, u_0)}{N_n^{(e)}} (a^{(e),(\pm)} \times \right. \\ \times Re_n^{(1)'}(\pm c, u) + b^{(e),(\pm)} Re_n^{(4)'}(\pm c, u)) Se_n(c, v_0) \times \\ \times Se_n(c, v) + \frac{Ro_n^{(1)'}(c, u_0)}{N_n^{(o)}} (a^{(o),(\pm)} Ro_n^{(1)'}(\pm c, u) + \\ \left. + b^{(o),(\pm)} Ro_n^{(4)'}(\pm c, u)) So_n(c, v_0) So_n(c, v) \right]. \quad (21)$$

The scattered magnetic field can be expressed as:

$$E_v^{s,m} = \frac{4jZ_0}{c\sqrt{\xi^2 - \eta^2}} \sum_{n=0}^{\infty} \left[ \frac{c_n^{(e),m}}{N_n^{(e)}} Re_n^{(1)'}(c, u_0) \right. \\ \times Re_n^{(4)'}(c, u) Se_n(c, v_0) Se_n(c, v) + \frac{c_n^{(o),m}}{N_n^{(o)}} \times \\ \times Ro_n^{(1)'}(c, u_0) Ro_n^{(4)'}(c, u) So_n(c, v_0) So_n(c, v) \left. \right]. \quad (22)$$

Approximation of far is applied when  $\xi \rightarrow \infty$

$$Re, o_n^{(4)}(c, \xi) \approx \frac{j^n}{\sqrt{c\xi}} e^{-jc\xi + j\frac{\pi}{4}} \\ \approx \frac{j^n}{\sqrt{kp}} e^{-jc\xi + j\frac{\pi}{4}} \quad (23)$$

Where  $p = \sqrt{x^2 + y^2}, p|_{\infty} \approx \frac{d}{2}\xi$ ; where  $\xi = \cosh(u)$ .

Then, the scattered magnetic far field can be written as:

$$H_z^{s,m}|_{\xi \rightarrow \infty} \approx \frac{e^{-jkp}}{\sqrt{kp}} e^{j\frac{\pi}{4}} 4 \sum_{n=0}^{\infty} j^n \left[ \frac{c_n^{(e),m}}{N_n^{(e)}} Re_n^{(1)'}(c, u_0) \times \right. \\ \times Se_n(c, v_0) Se_n(c, \cos\varphi) + \frac{c_n^{(o),m}}{N_n^{(o)}} Ro_n^{(1)'}(c, u_0) \times \\ \left. \times So_n(c, v_0) So_n(c, \cos\varphi) \right]. \quad (24)$$

The solution for even mode is provide, and that for odd mode is obtained by replacing  $Re_n^{(1),(4)}$  and their derivatives with  $Ro_n^{(1),(4)}$  and their derivatives. Then the expansion coefficients are expressed as:

$$a^{(e),(\pm)} = \frac{\mp \zeta_1 Re^{(4)'}(\pm c, u_1)\alpha}{\Delta(\pm)}, \quad (25)$$

$$b^{(e),(\pm)} = \frac{\pm \zeta_1 Re^{(1)'}(\pm c, u_1)\alpha}{\Delta(\pm)}, \quad (26)$$

$$c^{(e),(\pm)} = -\frac{1}{\Delta(\pm)} [Re^{(1)}(c, u_2) Re^{(1)'}(c, u_0) \times \\ \times Re^{(4)}(c, u_0) \Delta_1(\pm) \pm \zeta_1 Re^{(1)}(c, u_0) Re^{(1)'}(c, u_2) \times \\ \times Re^{(4)}(c, u_0) \Delta_2(\pm)], \quad (27)$$

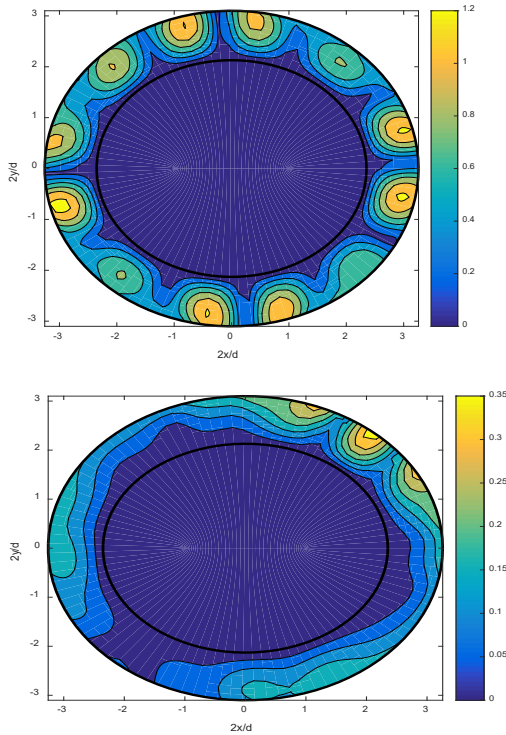
$$\alpha = Re^{(1)}(c, u_0)Re^{(1)'}(c, u_2)Re^{(4)}(c, u_2) \times \\ \times Re^{(4)'}(c, u_0) - Re^{(1)}(c, u_2)Re^{(1)'}(c, u_0) \times \\ \times Re^{(4)}(c, u_0)Re^{(4)'}(c, u_2) \quad , \quad (28)$$

$$\Delta_1(\pm) = Re^{(1)'}(\pm c, u_1)Re^{(4)'}(\pm c, u_2) - \\ - Re^{(1)'}(\pm c, u_2)Re^{(4)'}(\pm c, u_1) \quad , \quad (29)$$

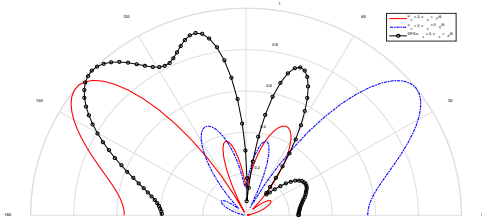
$$\Delta_2(\pm) = Re^{(1)}(\pm c, u_2)Re^{(4)'}(\pm c, u_1) - \\ - Re^{(1)'}(\pm c, u_1)Re^{(4)}(\pm c, u_2) \quad , \quad (30)$$

Parameter  $\Delta$  is retrieved as:

$$\Delta(\pm) = Re^{(1)}(c, u_0)Re^{(1)'}(c, u_0)[Re^{(4)}(c, u_2) \times \\ \times \Delta_1(\pm) \pm \zeta_1 Re^{(4)'}(c, u_2)\Delta_2(\pm)] \quad (31)$$



**Fig.4.** Comparison of behavior of  $|E_z|$  when electric line source is located at  $u_0 = 2, v_0 = \pi/6, u_1 = 1, u_2 = 1.85, \delta = 2$ : (a) DPS coating and (b) DNG coating



**Fig. 5.** Effect of the coating layer dimension and material properties on magnetic far field pattern of magnetic dipole from the structure when being coated by DPS and DNG, where  $\zeta = 0.5$ .

### 3. Numerical analysis

In figure 4, near field pattern in the area inside the coating layer is shown when electric line source is located at  $u_0 = 2, v_0 = \pi/6, u_1 = 1, u_2 = 1.85$ , all the quantities are normalized to  $\lambda$ , material property  $\delta = 2$ . It can be seen that the field trapped in DPS is much more of that in the case of DNG and more equally distributed in the structure. In Figure 5, all the quantities are normalized with reference to circular cylindrical coordinates  $(\rho, \varphi, z)$ . In order to validate the proposed computational scheme, two magnetic dipoles are placed symmetrically to  $-y$  axis, when dipole 1:  $u_1 = 2, v_1 = \pi/6$  and dipole 2:  $u_2 = 2, v_2 = 5\pi/6$ . Such that, scattered far field of dipole 1 (red solid line) and dipole 2 (blue dash-dot line) are exactly symmetric to  $-y$  axis. When changing the coating layer for the case of Dipole 1 to DPS, scattered magnetic field  $H_\theta$  is represented in marked black line, with the pattern is shifted toward the position of dipole.

### 4. Conclusion

For this particular geometry, with hollow and infinite structures, commercial simulator cannot always provide exact solution. In order to tackle this issue, fields in elliptical cylinder coordinate are derived. The structure in this paper is worth investigating because it contains sharp edges of metallic core, hollow and infinite bodies of layers. Analytical solutions for this geometry can be used as reference to validate the accuracy of the other electromagnetic solvers.

### Appendix A. Mathieu's functions and properties

Regarding computational cost and accuracy of this boundary-value problem, all the fields are represented in a closed form of asymptotic expression. In this care, the infinity is restricted to twenty-five terms of summation to achieve an error less than one percent. This fact means that if the field is calculated as twenty-five terms of summation, the absolute difference is less than one percent. Radial Mathieu's functions of the third kind and fourth kind in even mode can be given as:

$$Re_n^{(3)}(c, u) = Re_n^{(1)}(c, u) + iRe_n^{(2)}(c, u)$$

$$Re_n^{(4)}(c, u) = Re_n^{(1)}(c, u) - iRe_n^{(2)}(c, u)$$

And also for the odd mode:

$$Ro_n^{(3)}(c, u) = Ro_n^{(1)}(c, u) + iRo_n^{(2)}(c, u)$$

$$Ro_n^{(4)}(c, u) = Ro_n^{(1)}(c, u) - iRo_n^{(2)}(c, u)$$

It is also worth pointing out that the scheme of Mathieu's functions by Jiangmin Jin [8] and Erricolo [6] which have  $q = \frac{k^2 d^2}{16}$ . This research carried out in

this context implements the dimensionless parameter  $c = \frac{kd}{2}$ , such that  $c = \frac{q^2}{4}$ . Radial functions follow the Wronskian relation for both even mode and odd mode in the both DPS ( $c$ ) and DNG ( $-c$ ) material.

$$\text{Re}, o^{(1)} \frac{\partial \text{Re}, o^{(2)}}{\partial u} - \text{Re}, o^{(2)} \frac{\partial \text{Re}, o^{(1)}}{\partial u} = 1 \quad (32)$$

## References

- [1] T. Negishi, D. Erricolo and P. L. E. Uslenghi, Metamaterial Spheroidal Cavity to Enhance Dipole Radiation, in IEEE Transactions on Antennas and Propagation, vol. 63, no. 6, pp. 2802-2807, June 2015.
- [2] O.Akgol, D. Erricolo and P. L. E. Uslenghi, Exact Imaging by an Elliptic Lens, in IEEE Antennas and Wireless Propagation Letters, vol. 10, pp. 639-642, 2011.
- [3] O. Akol, V. G. Daniele, D. Erricolo and P. L. E. Uslenghi, Radiation From a Line Source Shielded by a Confocal Elliptic Layer of DNG Metamaterial, in IEEE Antennas and Wireless Propagation Letters, vol. 10, pp. 943-946, 2011.
- [4] J.J Bowman, T.B.A Senior, and P.L.E Uslenghi, Electromagnetic and Acoustic Scattering by simple Shapes, Amsterdam: North Holland Publishing Co., 1969. Reprinted by Hemisphere Publishing Co., New York, 1987.
- [5] J.A Stratton Electromagnetic theory, New York: McGraw-Hill, 1941.
- [6] Danilo Erricolo and Giuseppe Carluccio, Algorithm 934: Fortran 90 subroutines to compute Mathieu functions for complex values of the parameter ACM Trans. Math. Softw. 40, 1, Article 8 (October 2013).
- [7] P. L. E. Uslenghi, Exact penetration, radiation, and scattering for a slotted semielliptical channel filled with isorefractive material, IEEE Trans. Antennas Propag., vol.52, no.6 pp.1473-1480, June 2004.
- [8] S.Zhang and J.M Jin Computation of Special Functions, New York: Wiley, 1996.