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Flatness-Based Nonlinear Control for Path Planning and Tracking of Sloshing Liquid Container

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Abstract

The movement of the liquid inside the container, known as sloshing, is usually undesired. Thus, there is the necessity to keep under control the peaks that the liquid free-surface exhibits during motion. This paper aims at providing a solution for suppressing sloshing liquid in horizontally moving cylindrical container. After introducing equivalent discrete models based on a mass-spring-damper system introduced by the literature (non-linear model), the identification and utilization of flat outputs is presented to generate rest-to-rest trajectories in sloshing liquid systems, which is ensure the equilibrium of the sloshing height at both initial and final points. Moreover, a sliding-mode controller is described to solve the trajectory tracking problem. The effectiveness of the proposed approach is demonstrated through numerical simulations comparisons with a model predictive controller (MPC). This research contributes to the advancement of control techniques for anti-sloshing technology systems, enabling enhanced stability, performance, and safety in various engineering applications.

Keywords: Sloshing, flat output, rest-to-rest trajectory, sliding-mode control, model predictive control

1. Introduction

Sloshing, which refers to the motion of liquid in a container, is a phenomenon encountered in various engineering systems such as liquid-filled industrial equipment, spacecraft, and automotive fuel tanks. The significant challenges in controlling and predicting the behavior of the liquid, thereby impacting the stability and performance of the overall system. To address these challenges, it is important to accurately estimate the movement of liquid inside the container to prevent the liquid from overflowing. Thus, modeling and control techniques are crucial.

In recent years, there have been many research papers on developing nonlinear models for sloshing liquids that capture the complex dynamics and nonlinearity associated with this phenomenon. A novel approach, based on the mass-spring-damper model [1], is proposed in [2] for the sloshing-height estimation. The generalized coordinates describing the system are the mass displacements from the reference position. This model provides a more realistic representation of sloshing behavior compared to linear models, enabling better prediction and control strategies. However, the control design for nonlinear sloshing systems remains a challenging task due to their complex dynamics. The purpose of control is to move the container to the desired position without causing the liquid to vibrate with a large amplitude, or at least to suppress the sloshing at the beginning and end of the point-to-point transfer. As a result, generating a trajectory that matches all the requirements becomes part of the problem. The mass-spring-damper model is validated for 1-dimensional motions in [2], and it is exploited in [3] and [4] to plan anti-sloshing trajectories. The same technique is used in a software application presented in [5] to execute simulations of liquid sloshing in cylindrical and rectangular containers.

The concept of "flatness" has emerged as a powerful tool for the analysis and control of nonlinear systems [6, 7]. Flatness theory aims to identify a set of variables called flat outputs, which can be explicitly expressed as a function of the system's state and control inputs [8-10]. By leveraging this property, the control problem can be transformed into a trajectory tracking problem for the flat outputs, simplifying the design and implementation of control strategies. [11] showed that the liquid sloshing suppression problem can be viewed as one of appropriate flat output trajectory planning. However, this approach still lacks high accuracy because it uses a linear model for simplicity, and hence tracking errors may appear.

The goal of this paper is to investigate the flat output characterization of sloshing liquid nonlinear models. The key variables that exhibit flatness

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properties in sloshing systems are defined, and their effect on the system's overall behavior is analyzed. Furthermore, the implications of flatness in the design of control strategies for sloshing liquid systems, including trajectory planning and tracking, are implemented by using sliding mode control (SMC), which is known to be robust [12] and can be used to yield robust performance against model inaccuracy and uncertainties. It is one of the best-known and most effective robust control methods for nonlinear uncertain systems. This elegant approach has been intensively developed and applied to a wide spectrum of system types, see, for example, [13-18]. The ability to globally stabilize the system and its inherent insensitivity to a class of disturbance signals are core properties of the theory.

The paper is structured as follows: Section 2 presents the liquid sloshing non-linear model borrowed from the work of and the equations of motion (EOMs) in terms of the corresponding generalized coordinates. In Section 3, the flatness of the adopted non-linear model is established, and the sloshing suppression problem as a rest-to-rest transfer problem for the flat output is clarified. A sliding-mode controller is developed in Section 4. The effectiveness of the proposed approach through numerical simulations and comparisons with Model Predictive Controller (MPC) is shown in Section 5. Finally, in Section 6, conclusions are drawn and suggestions for future developments are given.

2. Mathematical Model

The sloshing estimation in liquid transfer is modelled in several recent studies. In this paper, we borrowed the non-linear model introduced, and experimentally justified in [2]. Liquid sloshing dynamics is modeled by mass-spring-damper shown by Fig. 1.

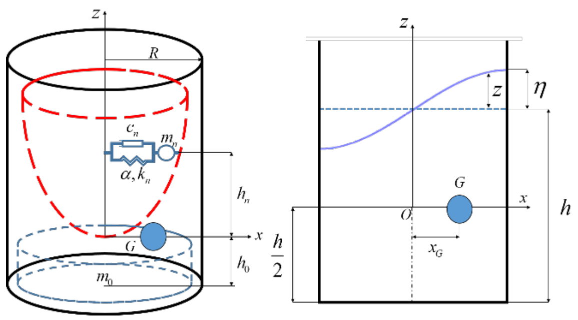

Fig. 1. Non-linear equivalent model and representation of the free surface of the liquid

2.1. Model Parameters

The moving container used in the model is cylindrical type with radius R, filled with a liquid of height h and mass m_F. The equivalent discrete model used to describe the motion of the free surface is a mass-spring-damper model comprises a rigid mass m_0 (whose signed vertical distance from the liquid's centre of gravity G is h_0) that moves rigidly with the container, and a series of moving masses m_n, with each one of them representing the equivalent mass of a sloshing mode (Fig. 1). Each modal mass m_n is restrained by a spring k_n and a damper c_n, and its signed vertical distance from G is h_n.

The model parameters are provided in [20] as follows:

$$m_n = m_F \frac{2R}{\xi_{1n} h(\xi_{1n}^2 - 1)} \tanh\left(\xi_{1n} \frac{h}{R}\right) \tag{1}$$

$$\omega_n^2 = \frac{k_n}{m_n} = g \frac{\xi_{1n}}{R} \tanh\left(\xi_{1n} \frac{h}{R}\right) \tag{2}$$

In (1), (2), ξ_{1n} is a constant parameter known for every sloshing mode, which represent the root of the derivative of the Bessel function of the first kind with respect to the radial coordinate r, for the 1st circumferential mode and the nth radial mode, and g is the gravity acceleration.

The damping ratio depends on the liquid height h, liquid kinematic viscosity υ, and tank diameter R [21]:

$$\zeta = \frac{2.89}{\pi} \sqrt{\frac{\upsilon}{R^{3/2} g^{1/2}}} \tag{3}$$

2.2. The Equation of the Free Surface

In this non-linear model, the sloshing mass m_n slides on a parabolic surface. The parabolic surface's analytical formula (Fig. 1) is given as follow:

$$z_n = \frac{C_n}{2R} x_n^2 \tag{4}$$

where $C_n = \omega_n^2 R/g$ and the time derivative of (4) gives:

$$\dot{z}_n = \frac{C_n}{R} x_n \dot{x}_n \tag{5}$$

The equation of motion can be obtained by the Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_n}\right) - \frac{\partial T}{\partial x_n} + \frac{\partial V}{\partial x_n} = -\frac{\partial D}{\partial x_n} \tag{6}$$

where:

T is the kinetic energy of the nth sloshing mass:

$$T = \frac{1}{2} m_n [(\dot{x}_0 + \dot{x}_n)^2 + \dot{z}_n^2] \tag{7}$$

V is the potential energy considers the contribution of gravity and the non-linear-spring forces:

$$V = m_n g z + \int_0^{r_n} \alpha_n k_n r_n^{2w-1} dr_n \tag{8}$$

D is the Rayleigh function accounts for energy dissipation:

$$D = \frac{1}{2} c_n (\dot{x}_n^2 + \dot{z}_n^2) \quad (9)$$

Substitute (7), (8) and (9) to (6), we obtain the equation of motion for the non-linear model:

$$(1 + \beta x_n^2)\ddot{x}_n + (2\omega_n \zeta_n (1 + \beta x_n^2) + \beta x_n \dot{x}_n)\dot{x}_n + \omega_n^2 \left(1 + \alpha \frac{x_n^{2w-2}}{R^2}\right) x_n + \ddot{x}_0 = 0 \quad (10)$$

where $\beta = \omega_n^4/g^2$, ω_n and ζ_n are provided in (2) and (3). Constant α is a dimensionless constant of the non-linear spring. We choose the value of w and α the same as in [2]: $w = 2, \alpha = 0.58$.

2.3. Sloshing-Height Estimation

According to [21], the planar function describes the shape of the liquid free surface:

$$z(r, \theta, \bar{\eta}_n) = \sum_n \bar{\eta}_n \frac{J_1\left(\xi_{1n} \frac{r}{R}\right)}{J_1(\xi_{1n})} \cos(\theta) \quad (11)$$

where J_1 is the Bessel function of order 1. $\bar{\eta}_n$ is the sloshing height at the container wall in the motion plane associated with the *n*-th mode.

From Fig. 1, consider the conservation of the centre of gravity x-coordinate between the continuum model and the equivalent model:

$$x_G m_f = \sum_n x_n m_n \quad (12)$$

The position of the centre of mass x_G is:

$$x_G = \frac{1}{\pi R^2 h} \int_0^R \int_0^{2\pi} \int_{-\frac{h}{2}}^{\frac{h}{2} + z(r,\theta,\bar{\eta}_n)} r^2 sin\theta dz d\theta dr = \frac{R}{h} \sum_n \frac{\bar{\eta}_n}{\xi_{1n}^2} \quad (13)$$

Finally, the sloshing height estimation for non-linear model is obtained by substitute (11), (13) to (12):

$$\eta = \frac{\xi_{1n}^2 h m_n}{m_F R} x_n = \gamma x_n \quad (14)$$

3. Flat Output Characterizations

3.1. Preliminaries

Consider a general system:

$$\dot{\boldsymbol{x}}(t) = F(\boldsymbol{x}(t), \boldsymbol{u}(t)) \quad (15)$$

with $\boldsymbol{x}(t) \in \mathbb{R}^n$ and $\boldsymbol{u}(t) \in \mathbb{R}^m$ are the state vector and the input vector, respectively. The nonlinear system (15) is called differentially flat if there exists a set of independent variables referred to as flat output $\boldsymbol{y}(t) \in \mathbb{R}^m$:

$$\boldsymbol{y}(t) = h_0\left(\boldsymbol{x}(t), \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t), \dots, \boldsymbol{u}^{(q)}(t)\right) \quad (16)$$

such that every other system variable (including the input variables) is a function of the flat output and a finite number of its successive time derivatives.

$$\boldsymbol{x}(t) = h_1\left(\boldsymbol{y}(t), \dot{\boldsymbol{y}}(t), \dots, \boldsymbol{y}^{(q)}(t)\right) \quad (17.1)$$

$$\boldsymbol{u}(t) = h_2\left(\boldsymbol{y}(t), \dot{\boldsymbol{y}}(t), \dots, \boldsymbol{y}^{(q+1)}(t)\right) \quad (17.2)$$

Remark 1. *A system's flatness and controllability characteristics are closely connected. A linear system is verified to be flat if and only if it is controllable. Furthermore, the number of flat outputs is equal to the number of inputs for any system which accepts a flatness-based representation.*

3.2. Flat Output of the Nonlinear Sloshing Model

Set $\beta = \omega_n^4/g^2$, the equation of motion (10) can be rewritten as follows:

$$M(x_n)\ddot{x}_n + r(x_n, \dot{x}_n)\dot{x}_n + k(x_n)x_n = -\ddot{x}_0 \quad (18)$$

where:

$$M(x_n) = 1 + \beta x_n^2 \quad (19)$$

$$r(x_n, \dot{x}_n) = 2\omega_n \zeta_n M + \beta x_n \dot{x}_n \quad (20)$$

$$k_x(x_n) = \omega_n^2 \left(1 + \frac{\alpha}{R^2} x_n^{2(w-1)}\right) \quad (21)$$

$$u = \ddot{x}_0 \quad (22)$$

According to [11], we can choose the flat output in the form of:

$$y = x_0 - \frac{r}{k_x}\dot{x}_0 + \left(M - \frac{r^2}{k_x}\right) x_n - \frac{Mr}{k_x}\dot{x}_n \quad (23)$$

To express the states x_n, x_0, η and input u as a function of y:

$$x_n = -\frac{1}{k_x}\ddot{y} = -\frac{1}{\omega_n^2\left(1 + \frac{\alpha}{R^2} x_n^{2(w-1)}\right)}\ddot{y} \quad (24)$$

$$x_0 = -\frac{1}{k_x}(M\ddot{y} + r\dot{y} + k_x y) \quad (25)$$

$$u = -(M\ddot{x}_n + r\dot{x}_n + k_x x_n) \quad (26)$$

$$\eta = -\frac{\gamma}{k_x}\ddot{y} \quad (27)$$

Since $w = 2$, (24) becomes a 3-degree equation:

$$\omega_n^2 \frac{\alpha}{R^2} x_n^3 + \omega_n^2 x_n + \ddot{y} = 0 \quad (28)$$

with only one solution:

$$x_n = \frac{R}{\sqrt{3\alpha}}\left(\sqrt[3]{k+\sqrt{k^2+1}}+\sqrt[3]{k-\sqrt{k^2+1}}\right) = f_0(\ddot{y}) \tag{29}$$

with: $k(\ddot{y}) = -\frac{3\sqrt{3\alpha}}{2R\omega_n^2}\ddot{y}$.

The first and second order derivatives of x_n are:

$$\dot{x}_n = \frac{R}{\sqrt{3\alpha}}\frac{\sqrt[3]{k+\sqrt{k^2+1}}-\sqrt[3]{k-\sqrt{k^2+1}}}{3\sqrt{k^2+1}}k\left(y^{(3)}\right) = f_1\left(\ddot{y}, y^{(3)}\right) \tag{30}$$

$$\ddot{x}_n = \frac{R}{\sqrt{3\alpha}}\left(\frac{\sqrt[3]{k+\sqrt{k^2+1}}-\sqrt[3]{k-\sqrt{k^2+1}}}{3\sqrt{k^2+1}}k\left(y^{(4)}\right) + \left(\frac{\sqrt[3]{k-\sqrt{k^2+1}}+\sqrt[3]{k+\sqrt{k^2+1}}}{9(k^2+1)} + \frac{k\left(\sqrt[3]{k-\sqrt{k^2+1}}-\sqrt[3]{k+\sqrt{k^2+1}}\right)}{3\sqrt{(k^2+1)^3}}\right)k\left(y^{(3)}\right)\right) = f_2\left(\ddot{y}, y^{(3)}, y^{(4)}\right) \tag{31}$$

Substitute (29), (30) to (19), (20) and (21), we obtain:

$$M(\ddot{y}) = 1 + \frac{C_n^2}{R^2}\left(f_0(\ddot{y})\right)^2 \tag{32}$$

$$r\left(\ddot{y}, y^{(3)}\right) = 2\omega_n\zeta_n M(\ddot{y}) + \frac{C_n^2}{R^2}f_0(\ddot{y})f_1\left(\ddot{y}, y^{(3)}\right) \tag{33}$$

$$k_x(\ddot{y}) = \omega_n^2\left(1 + \frac{\alpha}{R^2}\left(f_0(\ddot{y})\right)^2\right) \tag{34}$$

and yields:

$$x_0 = -\frac{1}{k_x(\ddot{y})}\left(M(\ddot{y})\ddot{y} + r\left(\ddot{y}, y^{(3)}\right)\dot{y} + k(\ddot{y})y\right) \tag{35}$$

$$u = -\left(M(\ddot{y})f_2\left(\ddot{y}, y^{(3)}, y^{(4)}\right) + r\left(\ddot{y}, y^{(3)}\right)f_1\left(\ddot{y}, y^{(3)}\right) + k_x(\ddot{y})f_0(\ddot{y})\right) \tag{36}$$

which proves that all state can be expressed as functions of y and a finite number of its derivatives, and thus the system (10) is flat with y as the flat output.

3.3. Rest-to-Rest Trajectory Planning

Traditionallly, rest-to-rest trajectory planning does not consider the internal dynamics. However in this study, since all the system variables can be expressed as functions of this flat output and a finite number of its successive derivatives, the sloshing height η can be ensured to reach a stationary condition when the end point is attained.

We want to generate displacements of the stage from one steady state to another one with the base also in steady state at the stage's final position. In rest positions, it suffices to generate a polynomial trajectory for y with respect to time, interpolating the initial and final conditions:

$$x_0(t_0) = x_0,\quad \dot{x}_0(t_0) = 0,\quad x_n(t_0) = 0,\quad \dot{x}_n(t_0) = 0,\quad u(t_0) = 0$$

$$x_0(t_1) = x_1,\quad \dot{x}_0(t_1) = 0,\quad x_n(t_1) = 0,\quad \dot{x}_n(t_1) = 0,\quad u(t_1) = 0$$

Thus

$$y_0(t_0) = x_0,\quad \dot{y}_0(t_0) = 0,\quad \ddot{y}(t_0) = 0,\quad y^{(3)}(t_0) = 0,\quad y^{(4)}(t_0) = 0$$

$$y_0(t_1) = x_1,\quad \dot{y}_0(t_1) = 0,\quad \ddot{y}(t_1) = 0,\quad y^{(3)}(t_1) = 0,\quad y^{(4)}(t_1) = 0$$

Since there are 10 initial and final conditions, the minimal degree of an interpolating polynomial is equal to 9. According to [19], we get the interpolation polynomials:

$$y = x_0 + (x_1 - x_0)(126 - 420\tau + 540\tau^2 - 315\tau^3 + 70\tau^4)\tau^5 \tag{37}$$

with $\tau = (t - t_0)/(t_1 - t_0)$. To maintain smoothness and stability at the start and end points, it is essential to set all derivatives of y to zero. To achieve this, it is possible to introduce extra null derivatives of order four or higher at both the start and end points. This approach helps prevent oscillations or unstable behaviours at the end point. One current drawback of the method is that the length of time the trajectory will take must be known a prior to satisfy the initial conditions.

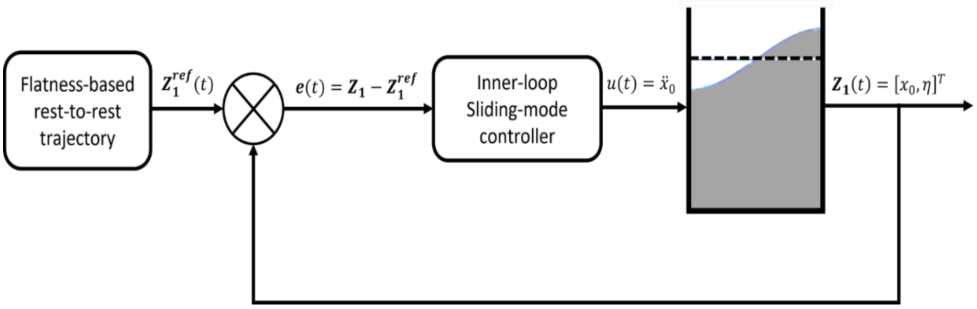

Fig. 2. Control Scheme for Sloshing Liquid Containers

In practice, it has been observed that trajectories with inappropriate time and distance can cause the nonlinear sloshing model to exceed the safe domain. Therefore, it is important to choose reasonable durations and distances to avoid strongly nonlinear motion, where the liquid free-surface exhibits instantaneous peaks, characterized by swirling shapes.

The corresponding solution is then stored as the reference trajectory used for the trajectory controller which will be designed in the next section. The control structure is shown in Fig. 2.

4. Sliding Mode Controller Design

Consider the non-linear sloshing system with output variable $\boldsymbol{Z_1} = [x_0\ \eta]^T$. We need to make output $\boldsymbol{Z_1}$ track the desired trajectory $\boldsymbol{Z_{1d}} = [x_{0d}\ \eta_d]^T$, which designed through flat output y in the previous section.

Define

$$\boldsymbol{Z_2} = [\dot{x}_0\ \dot{\eta}]^T \tag{38}$$

Equation (18) is rewritten as follow:

$$\dot{\boldsymbol{Z}}_{\boldsymbol{1}} = \boldsymbol{Z_2} \tag{39}$$

$$\dot{\boldsymbol{Z}}_{\boldsymbol{2}} = -\boldsymbol{f} + \boldsymbol{g}u \tag{40}$$

where:

$$\boldsymbol{f} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{k_x}{M} \end{bmatrix} \boldsymbol{Z_1} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{r}{M} \end{bmatrix} \boldsymbol{Z_2} \tag{41}$$

$$\boldsymbol{g} = \begin{bmatrix} 1 \\ -\gamma/M \end{bmatrix} \tag{42}$$

Define $\boldsymbol{e} = \boldsymbol{Z_1} - \boldsymbol{Z_{1d}}$ as the tracking error. Then the sliding surface is design as:

$$\boldsymbol{s} = \boldsymbol{\lambda e} + \dot{\boldsymbol{e}} \tag{43}$$

where λ is a designed positive definite diagonal matrix. Take the time derivative we obtain that:

$$\dot{\boldsymbol{s}} = \boldsymbol{\lambda}\dot{\boldsymbol{e}} + \ddot{\boldsymbol{e}} = \boldsymbol{\lambda}\dot{\boldsymbol{e}} + \ddot{\boldsymbol{Z}}_{\boldsymbol{1}} - \ddot{\boldsymbol{Z}}_{\boldsymbol{1d}} = \boldsymbol{\lambda}\dot{\boldsymbol{e}} + \left(-\boldsymbol{f} + \boldsymbol{g}u - \ddot{\boldsymbol{Z}}_{\boldsymbol{1d}}\right) \tag{44}$$

The sliding mode controller is proposed as follows:

$$u = \boldsymbol{g}^{-\boldsymbol{1}}\left(\boldsymbol{f} + \ddot{\boldsymbol{Z}}_{\boldsymbol{1d}} - \boldsymbol{\lambda}\left(\boldsymbol{Z_2} - \dot{\boldsymbol{Z}}_{\boldsymbol{1d}}\right) - \boldsymbol{k_1 s} - \boldsymbol{k_2} sign(\boldsymbol{s})\right) \tag{45}$$

where $\boldsymbol{k_1}$, $\boldsymbol{k_2}$ is the positive definite switching gain matrices. $\boldsymbol{g}^{-\boldsymbol{1}} = \left[\frac{M^2}{\gamma^2 + M^2}, -\frac{\gamma M}{\gamma^2 + M^2}\right]$ is pseudoinverse matrix of $\boldsymbol{g}$. Select the Lyapunov function as

$$L = \frac{1}{2}\boldsymbol{s^T s} \tag{46}$$

Therefore, we have:

$$\begin{aligned} \dot{L} = \boldsymbol{s^T}\dot{\boldsymbol{s}} &= \boldsymbol{s}^T\left[\boldsymbol{\lambda}\dot{\boldsymbol{e}} + \left(-\boldsymbol{f} + \boldsymbol{g}u - \ddot{\boldsymbol{Z}}_{\boldsymbol{1d}}\right)\right] \\ &= \boldsymbol{s}^T\left(-\boldsymbol{k_1 s} - \boldsymbol{k_2} sign(\boldsymbol{s})\right) \\ &\le -\lambda_{min}(\boldsymbol{k_1})\|\boldsymbol{s}\|^2 - \lambda_{min}(\boldsymbol{k_2})\|\boldsymbol{s}\| \\ &\le 0 \left(\dot{L} = 0 \iff \boldsymbol{s} = \boldsymbol{0}_{2\times1}\right) \end{aligned} \tag{47}$$

where $\lambda_{min}(\cdot)$ represents the minimum eigenvalue of a matrix.

5. Simulation results and Comparisons

In this Section, we first present simulation results of our proposed approach introduced in Section 3, 4. Then, various comparisons of our contributions with other approaches are provided.

5.1. Flat output characterization results

In this section, we carry out some simulations with a cylindrical container and the chosen liquid is water with the dynamic viscosity υ and density ρ, respectively. All the parameters used in the paper are given in Table 1.

Table 1. Parameters of Sloshing Liquid Simulation

Parameter	Value	Unit
R	50	mm
h	70	mm
g	9.81	m/s^2
ρ	997	kg/m^3
υ	0.001	Pa·s
α	0.58	–
w	2	–
ξ_{1n}	1.841	–

The trajectories are planned so that the container will move from initial point $x_0 = 0$ to the final point $x_i = 1.5$ in $T = 10$ [s]. A 2-1-2 trajectory will be used to compared with the flatness-based trajectory. The input acceleration is set as:

$$u_{0(2-1-2)}(t) = \begin{cases} 0.1 & \forall\ 1 < t \le 4 \\ -0.1 & \forall\ 6 < t \le 9 \\ 0 & otherwise \end{cases} \tag{48}$$

The rest-to-rest trajectory designed in Section 3 is shown in Fig. 2. We then deduce the corresponding trajectories for $x_0, \dot{x}_0, x_n, \eta, u$ and their comparisons as depicted in Fig. 3-7.

As shown in Fig. 3, both trajectories guide the containers to reach their destination points in a defined time. Although the 2-1-2 trajectory has a lower velocity and acceleration input, which is shown in Fig. 4, 5, the maximum sloshing height sometimes reaches outside the 0.6 [mm] range, which may cause the overflow phenomenon. Also the sloshing height does not reach zero when the container stops at the final point, whereas the rest-to-rest trajectory ensures that the equilibrium of the sloshing height at both initial and final points is equal to 0 during the simulation time. This state does not reach zero throughout the simulation time but is bound in an allowed range.

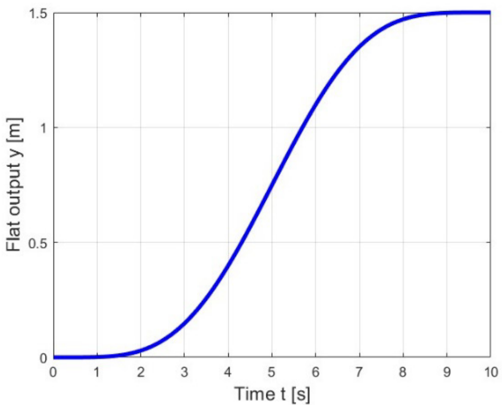

Fig. 3. Rest-to-rest Flat Output Trajectory

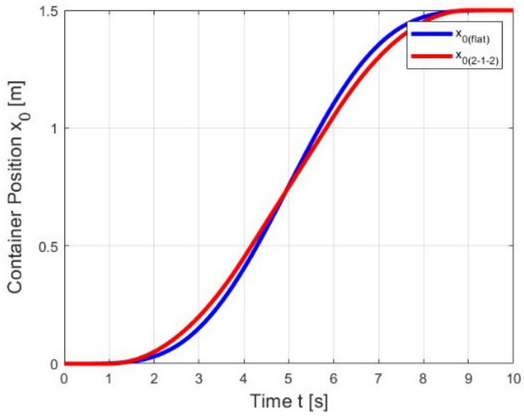

Fig. 4. Container Position

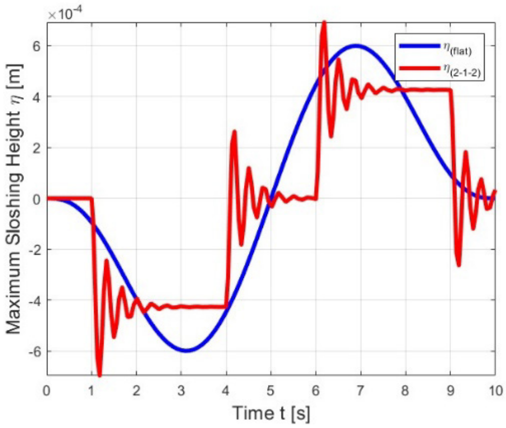

Fig. 5. Maximum Sloshing Height

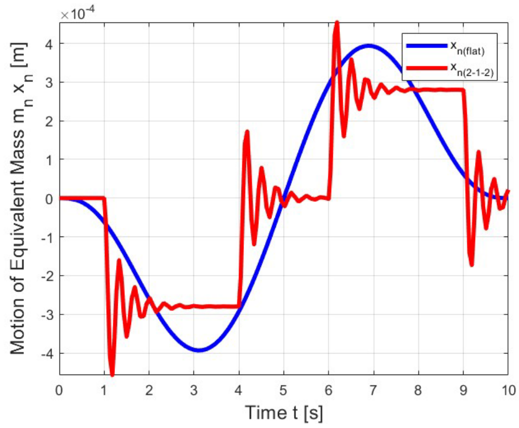

Fig. 6. Motion of Equivalent Mass

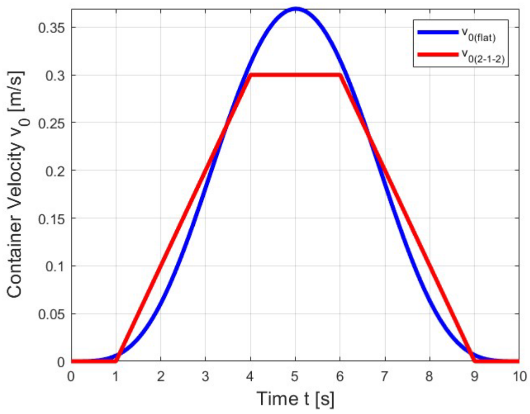

Fig. 7. Container Velocity

5.2. Controller Comparison

In this section, the tracking performance between SMC and MPC controller of sloshing liquid will be compared. In practice, systems have various physical limits such as response time, designed operating capacity, and more. Hence, Nonlinear Model Predictive Control (NMPC) is widely researched and applied. The advantage of NMPC lies in its ability to explicitly consider constraints, especially in complex systems that require meeting numerous conditions. This makes it a valuable approach for controlling complex systems with multiple constraints, providing a more effective and robust control solution. Furthermore, based on the prediction horizon, NMPC can show an optimal solution to optimize the future system behavior and feedforward disturbance compensation can be easily integrated into NMPC formulation.

To guarantee the operation of the system, we will consider the maximum acceleration of the system. And in this study, we use the *Nonlinear MPC Toolbox* integrated in *MATLAB-Simulink* to simulation. But the proposed general NMPC scheme is obtained by solving the following optimal control problem:

Minimize

$$J = \sum_{k=0}^{N_p} \|\boldsymbol{we}(k+j)\|^2 + \sum_{k=0}^{N_p} \|\lambda u\|^2 \quad (49)$$

Subject to

$$|u| \le u_{max}$$

where the weighted matrix is $\boldsymbol{w} = [180\ 180]$, $\lambda = 0.1$, and the predicted error output matrix is $\boldsymbol{e} = \boldsymbol{Z_{1d}}(k+j) - \boldsymbol{Z_1}(k+j)$ at sampling $(k+j)$.

These two inequality constraints are set the upper and the lower bound of the control acceleration, ensuring the optimal control solutions are admissible.

Comparative results in Fig. 9-12 indicate that both controllers have good tracking performance, but the SMC has higher tracking accuracy than the MPC. Both the container position and the sloshing height of the liquid can robustly track the planned trajectory in Section 3 using the SMC and MPC. The control input is plotted in Fig. 8. There are still a few errors compared with the acceleration input in Fig. 5. But these errors are allowed to remain within a limited range.

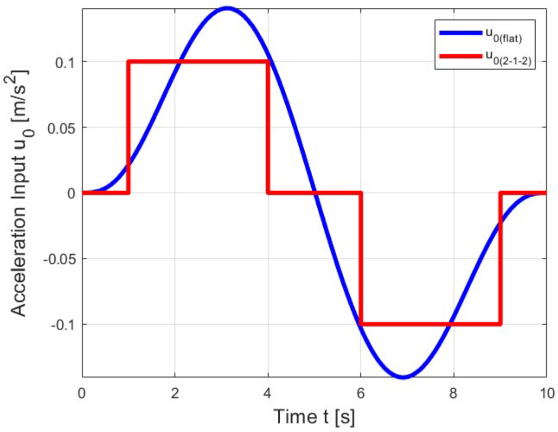

Fig. 8. Acceleration Input

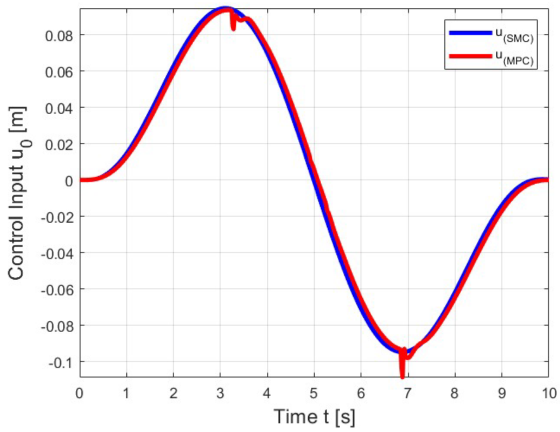

Fig. 9. Control Inputs

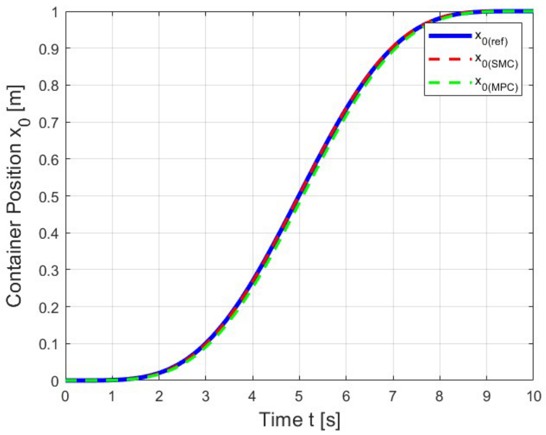

Fig. 10. Container Position Comparisons with Flat Reference

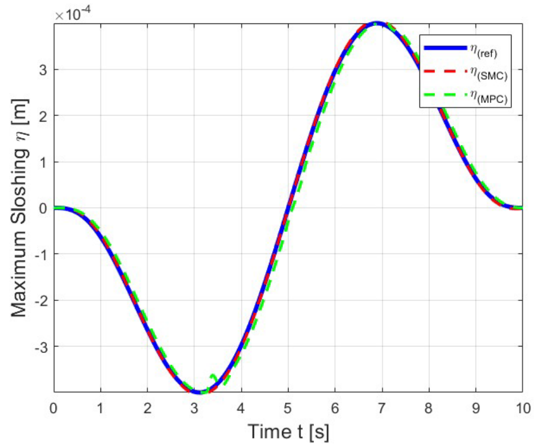

Fig. 11. Maximum Sloshing Height Comparisons

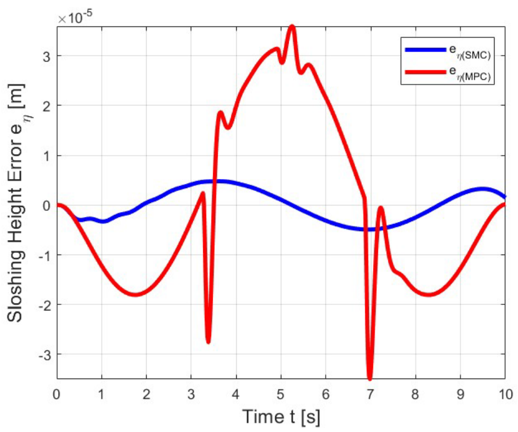

Fig. 12. Sloshing Height Errors

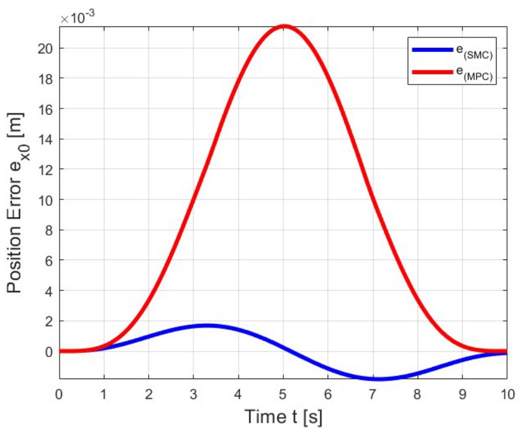

Fig. 13. Position Errors

6. Conclusion

In this paper, we have investigated the characterization of flat outputs for sloshing liquid nonlinear models to achieve rest-to-rest trajectories. Sloshing phenomena in liquid containers present significant challenges in engineering applications, and accurate control of sloshing dynamics is crucial for system stability and performance. By leveraging the concept of flatness theory, we have proposed a novel approach to identify key variables that exhibit flatness properties in sloshing systems. Next two control strategies are provided and compared to obtain robust trajectory tracking for container position and liquid sloshing height. Future work will expand the proposed flat output approach for 2-dimensional motion of liquid container and use an observer to obtain unmeasured state for controller.

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