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Low Computational Cost Algorithms for Solving Variational Inequalities over the Fixed Point Set

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Abstract

The variational inequality problem has many important applications in the fields of signal processing, image processing, optimal control, and many others. In this paper, we discuss several extragradient-like algorithms for solving variational inequalities over the fixed point set of a nonexpansive mapping. The considered methods are based on some existing ones. Our algorithms use dynamic step-sizes, chosen based on information of previous steps and under the assumptions that the involving mapping is pseudomonotone and Lipschitz continuous, the sequence generated by our algorithms converges to the desired solution. Compared with the original extragradient algorithm, the new ones have an advantage: they do not require to compute any projection onto the feasible set. This feature helps to reduce the computational cost of our methods. Moreover, to implement the new algorithms, we do not need to know the Lipschitz constant of the involving mapping. Also, we present some numerical experiments to verify the efficiency of the new algorithms.

Keywords: Variational inequality, Lipschitz continuity, pseudomonotonicity, extragradient algorithm.

1. Introduction

Let C be a nonempty, closed and convex set in Euclidean space $\mathbb{R}^m$, $A : C \to C$ be a mapping. The variational inequality problem of A on C is:

To find $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0 \ \forall x \in C. \tag{1}$$

This problem is an important tool in economics, operations research, and mathematical physics. It includes many problems of nonlinear analysis in a unified form, such as optimization, fixed point problems, Nash equilibrium problems, saddle point problems. A lot of algorithms for solving this problem have been proposed. Among them, the Gradient projection algorithm is the simplest one:

$$\begin{cases} x^0 \in C, \\ x^{k+1} = P_C\big(x^k - \lambda A(x^k)\big), \end{cases} \tag{2}$$

where $P_C(.) : \mathbb{R}^m \to C$ is the metric projection from $\mathbb{R}^m$ onto C.

Under the assumptions that A is γ-strongly pseudomonotone and L-Lipschitz continuous on, $\lambda \in (0, 2\frac{\gamma}{L^2})$, the sequence $\{x^k\}$ generated by (2) converges linearly to the unique solution of the problem (1). If A is only monotone instead of being strongly pseudomonotone, the Gradient projection algorithm, in general, is not convergent. In this case, the Extragradient algorithm [1] is a typical one for solving (1):

$$\begin{cases} x^0 \in C, \\ y^{k+1} = P_C\big(x^k - \lambda A(x^k)\big), \\ x^{k+1} = P_C\big(x^k - \lambda A(y^{k+1})\big). \end{cases} \tag{3}$$

Under the conditions that A is pseudomonotone and L-Lipschitz continuous on, $\lambda \in (0, \frac{1}{L})$, algorithm (3) converges to a solution of (1). This algorithm has been investigated and developed by a lot of authors, see [2, 3]. However, it has two drawbacks: First, it requires to compute the projection onto C twice in each iteration. This increases the computational cost of the algorithm if C has a complicated form. Second, to implement (3), we need to know the Lipschitz constant L of A. In practice, this constant can be very difficult to calculate.

In some real-world models, the feasible set C may not be given in an explicit form. For example, in [4, 5], Iiduka considered the power control problem model of CDMA networks. This model leads to a variational inequality over the fixed point set of a nonexpansive mapping:

To find $x^* \in \mathrm{Fix}(T)$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0 \ \ \forall x \in \mathrm{Fix}(T), \tag{4}$$

Denote by $Sol(A, \mathrm{Fix}(T))$ the solution set of (4). To solve this problem, the author proposed the following Ergodic algorithm:

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$$\begin{cases} x^0 \in C, \\ x^{k+1} = T\big(x^k - \lambda_k A(x^k)\big), \\ z^k = \frac{\sum_{i=0}^k \lambda_i x^i}{\sum_{i=0}^k \lambda_i}. \end{cases} \quad (5)$$

Under the assumptions that A is monotone, $\sum_{i=0}^{\infty} \lambda_i = \infty$, $\sum_{i=0}^{\infty} \lambda_i^2 < \infty$ and

$$Sol\big(A, \mathrm{Fix}(T)\big) \subset \Omega :=$$

$$\{z \in \mathrm{Fix}(T) : \langle Ay^k, z - y^k \rangle \le 0 \ \ \forall k \ge k_0\}, \quad (6)$$

the sequence $\{x^k\}$ generated by (5) converges to a desired solution. Compared to (3), the Ergodic algorithm has a clear advantage: it does not require to compute any projection onto C. However, due to condition $\sum_{i=0}^{\infty} \lambda_i^2 < \infty$, the step size of (5) decreases very rapidly, and thus, slows down the convergence rate of this algorithm.

Motivated by the works in [6, 7], in this paper, we introduce two new algorithms for solving (4). Our algorithms are designed to inherit the advantages and overcome the disadvantages of the existing ones. Namely, in each iteration of the new algorithms, we do not need to compute any projection onto C. Also, the new algorithms do not require to know the Lipschitz constant L of the involving mapping. Moreover, the steps size λ_k in the new algorithms need not satisfy the condition $\sum_{i=0}^{\infty} \lambda_i^2 < \infty$. All these features help to reduce the computational cost and speed up our algorithms.

This paper is organized as follows. Section 2 presents some notations and preliminary results that will be used in the sequel. We introduce the algorithms and establish convergence analysis in Section 3. Finally, some numerical experiments are reported in Section 4.

2. Preliminaries

We present some notations and preliminary results, which will be used in the next sections. Interested readers can find more details in [1].

Let $x \in \mathbb{R}^m$, denote

$$P_C(x) := \operatorname{argmin}\{\|z - x\| : z \in C\}.$$

The mapping $P_C(.)$ is called the projection onto C. Since C is closed and convex, this mapping is well defined for every $x \in \mathbb{R}^m$.

It holds that [1]:

(i) $\|P_C(x) - P_C(y)\| \le \|x - y\|$ *for all* $x, y \in \mathbb{R}^m$;

(ii) $\langle y - P_C(x), x - P_C(x) \rangle \le 0$ *for all* $x \in \mathbb{R}^m$, $y \in C$.

A mapping $A : \mathbb{R}^m \to \mathbb{R}^m$ is said to be [8]:

1. *pseudomonotone* on $\mathbb{R}^m$ if for all $x, y \in \mathbb{R}^m$, we have

$$\langle A(y), x - y \rangle \ge 0 \Longrightarrow \langle A(x), x - y \rangle \ge 0.$$

2. γ-*strongly pseudomonotone* on $\mathbb{R}^m$ if there exist a constant $\gamma \in (0, \infty)$ such that for all $x, y \in \mathbb{R}^m$, we have

$$\langle A(y), x - y \rangle \ge 0 \Longrightarrow \langle A(x), x - y \rangle \ge \gamma \|x - y\|^2.$$

3. *L-Lipschitz continuous* on $\mathbb{R}^m$ if there exist a constant $L \in (0, \infty)$ such that for all $x, y \in \mathbb{R}^m$, we have

$$\|Ax - Ay\| \le L\|x - y\|.$$

If $L = 1$, then the mapping is called *nonexpansive*.

3. Main Results

In this section, we introduce and investigate two methods to solve variational inequalities over the fixed point set. The following conditions need to be satisfied in order to obtain the convergence theorems of the proposed algorithms.

Assumption 3.1. Consider (4) under the following assumptions:

(A1) The mapping A is pseudomonotone on $\mathbb{R}^m$;

(A2) The mapping A is Lipschitz continuous on $\mathbb{R}^m$ (with unknown modulus);

(A3) The mapping T is nonexpansive;

(A4) $Sol\big(A, \mathrm{Fix}(T)\big) \ne \emptyset$.

3.1. First algorithm

For solving (4), we propose the following algorithm:

Algorithm 1.

Step 0. Choose $x^{-1}, x^0, y^0 \in \mathbb{R}^m$; $\rho, \delta \in (0,1)$; $\lambda_{-1} \in (0, \infty)$. Set $k = 0$.

Step 1. Given λ_{k-1}, y^k, and x^k.

If $\lambda_{k-1}\|A(x^{k-1}) - A(y^k)\| \le \rho\|x^{k-1} - y^k\|$ **then** set $\lambda_k = \lambda_{k-1}$ **else** set $\lambda_k = \lambda_{k-1}\delta$. Compute

$$y^{k+1} = x^k - \lambda_k A(x^k)$$

$$z^{k+1} = x^k - \lambda_k A(y^{k+1})$$

$$x^{k+1} = \frac{1}{2}(z^{k+1} + T(z^{k+1})).$$

Step 2. Update $k := k + 1$ and GOTO **Step 1.**

As we can see, in Algorithm 1, we do not need to calculate any projection. Instead, we just compute the value of the mapping T once in each iteration. This feature greatly reduces the computation cost of the algorithm.

Theorem 3.2. *Suppose that Assumption 3.1 holds. Moreover, there exists a number $k_0 \geq 0$ such that (6) is satisfied. Then, the sequence $\{x^k\}$ generated by Algorithm 1 converges to a solution of (4).*

Proof. We divide the proof of Theorem 3.2 into second steps.

Claim 1: The sequence $\{x^k\}$ is bounded.

Obviously, we have $\lambda_{k+1} \leq \lambda_k$ for all $k \geq 0$. We will prove that there exists $\epsilon > 0$ satisfying $\lambda_k \geq \epsilon$ for all $k \geq 0$. Indeed, in the opposite case, i.e., $\lim_{k\to\infty} \lambda_k = 0$, there exists a subsequence $\{\lambda_{k_i}\} \subset \{\lambda_k\}$ such that

$$\lambda_{k_i-1}\|A(x^{k_i-1}) - A(y^{k_i})\| > \rho\|x^{k_i-1} - y^{k_i}\|.$$

Let L be the Lipschitz modulus of A, it holds that

$$\lambda_{k_i-1} > \rho\frac{\|x^{k_i-1} - y^{k_i}\|}{\|A(x^{k_i-1}) - A(y^{k_i})\|} \geq \frac{\rho}{L} \ \forall i \geq 0.$$

This contradicts the assumption that $\lim_{k\to\infty} \lambda_k = 0$. Thus, there exists a number $k_0 > 0$ satisfying $\lambda_k = \lambda_{k_0}$ and

$$\lambda_k\|A(x^k) - A(y^{k+1})\| \leq \rho\|x^k - y^{k+1}\| \quad \forall k \geq k_0. \tag{7}$$

Since $y^{k+1} = x^k - \lambda_k A(x^k)$, we have

$$\langle y^{k+1} - x^k, y^{k+1} - z\rangle = \lambda_k\langle A(x^k), z - y^{k+1}\rangle \quad \forall z \in \mathbb{R}^m. \tag{8}$$

Analogously, from the definition of z^{k+1}, we obtain

$$\langle z^{k+1} - x^k, z^{k+1} - z\rangle = \lambda_k\langle A(y^{k+1}), z - z^{k+1}\rangle \quad \forall z \in \mathbb{R}^m. \tag{9}$$

Thus,

$$\begin{aligned}
&\|z^{k+1} - z\|^2\\
&= \|y^{k+1} - z\|^2 - \|z^{k+1} - y^{k+1}\|^2\\
&+2\langle z^{k+1} - y^{k+1}, z^{k+1} - z\rangle\\
&= \|x^k - z\|^2 - \|z^{k+1} - y^{k+1}\|^2\\
&-\|y^{k+1} - x^k\|^2 + 2\langle z^{k+1} - y^{k+1}, z^{k+1} - z\rangle\\
&+2\langle y^{k+1} - x^k, y^{k+1} - z\rangle\\
&= \|x^k - z\|^2 - \|z^{k+1} - y^{k+1}\|^2\\
&-\|y^{k+1} - x^k\|^2 + 2\langle z^{k+1} - x^k, z^{k+1} - z\rangle\\
&+2\langle y^{k+1} - x^k, y^{k+1} - z^{k+1}\rangle.
\end{aligned} \tag{10}$$

Combining (8), (9), and (10) we obtain

$$\begin{aligned}
\|z^{k+1} - z\|^2 &= \|x^k - z\|^2 - \|z^{k+1} - y^{k+1}\|^2\\
&-\|y^{k+1} - x^k\|^2 + 2\lambda_k\langle A(y^{k+1}), -z^{k+1}\rangle\\
&+2\lambda_k\langle A(x^k), z^{k+1} - y^{k+1}\rangle\\
&= \|x^k - z\|^2 - \|z^{k+1} - y^{k+1}\|^2\\
&-\|y^{k+1} - x^k\|^2 + 2\lambda_k\langle A(y^{k+1}), z - y^{k+1}\rangle\\
&+2\lambda_k\langle A(x^k) - A(y^{k+1}), z^{k+1} - y^{k+1}\rangle.
\end{aligned} \tag{11}$$

Combining (7), and (11), for all $k \geq k_0$ and $z \in \mathbb{R}^m$, we get

$$\begin{aligned}
\|z^{k+1} - z\|^2 &\leq \|x^k - z\|^2 - \|x^{k+1} - y^{k+1}\|^2\\
&-\|y^{k+1} - x^k\|^2 + 2\lambda_k\langle A(y^{k+1}), z - y^{k+1}\rangle\\
&+2\lambda_k\|A(x^k) - A(y^{k+1})\|\|z^{k+1} - y^{k+1}\|\\
&\leq \|x^k - z\|^2 - \|z^{k+1} - y^{k+1}\|^2\\
&-\|y^{k+1} - x^k\|^2 + 2\lambda_k\langle A(y^{k+1}), z - y^{k+1}\rangle\\
&+2\rho\|x^k - y^{k+1}\|\|z^{k+1} - y^{k+1}\|\\
&\leq \|x^k - z\|^2 - (1-\rho)\|z^{k+1} - y^{k+1}\|^2\\
&-(1-\rho)\|y^{k+1} - x^k\|^2\\
&+ 2\lambda_k\langle A(y^{k+1}), z - y^{k+1}\rangle.
\end{aligned} \tag{12}$$

On the other hand, since T is nonexpansive, for all $t \in \text{Fix}(T)$, it holds that

$$\|Tz^{k+1} - t\|^2 = \|Tz^{k+1} - Tt\|^2 \leq \|z^{k+1} - t\|^2. \tag{13}$$

From the definition of x^{k+1}, it follows that $Tz^{k+1} = 2x^{k+1} - z^{k+1}$. Combining this and (13), we have

$$\|(2x^{k+1} - 2t) - (z^{k+1} - t)\|^2 \leq \|z^{k+1} - t\|^2$$

or equivalently,

$$\|x^{k+1} - t\|^2 \leq \langle x^{k+1} - t, z^{k+1} - t\rangle.$$

Using the equality

$$\langle a, b\rangle = \frac{1}{2}(\|a\|^2 + \|b\|^2 - \|a+b\|^2),$$

we have

$$\|x^{k+1} - t\|^2 \leq \frac{1}{2}(\|x^{k+1} - t\|^2 + \|z^{k+1} - t\|^2 - \|x^{k+1} - z^{k+1}\|^2),$$

or

$$\|x^{k+1} - t\|^2 \leq \|z^{k+1} - t\|^2 - \|x^{k+1} - z^{k+1}\|^2. \tag{14}$$

Combining (12) and (14), for all $t \in \text{Fix}(T)$, we have

$$\begin{aligned}
\|x^{k+1} - t\|^2 \leq \|x^k - t\|^2 &- \|x^{k+1} - z^{k+1}\|^2\\
&-(1-\rho)\|z^{k+1} - y^{k+1}\|^2\\
&+2\lambda_k\langle A(y^{k+1}), t - y^{k+1}\rangle
\end{aligned}$$

$$-(1-\rho)\|y^{k+1}-x^k\|^2. \quad (15)$$

In (15), letting $t = t^* \in \Omega := \{z \in \text{Fix}(T) : \langle A(y^k), z - y^k\rangle \leq 0 \ \ \forall k \geq k_1\}$, we have

$$\|x^{k+1}-t^*\|^2 \leq \|x^k - t\|^2 - \|x^{k+1}-z^{k+1}\|^2$$
$$-(1-\rho)\|z^{k+1}-y^{k+1}\|^2$$
$$-(1-\rho)\|y^{k+1}-x^k\|^2. \quad (16)$$

The sequence $\{x^k - t^*\}$ in nonincreasing, and henceforth, being nonnegative, it is convergent. Moreover, we have

$$\lim_{k\to\infty}\|x^k - y^k\| = \lim_{k\to\infty}\|y^k - z^{k+1}\|$$
$$= \lim_{k\to\infty}\|x^{k+1}-z^{k+1}\|$$
$$= \lim_{k\to\infty}\|x^{k+1}-x^k\| = 0. \quad (17)$$

It follows that the sequence $\{x^k\}$ is bounded, and hence so are $\{y^k\}$ and $\{z^k\}$. There exists a subsequence $\{x^{k_i}\} \subset \{x^k\}$ such that $x^{k_i} \to x^k$.

This is end of the proof of Claim 1.

Claim 2: $x^* \in Sol(A, \text{Fix}(T))$.

Since T is nonexpansive, we have

$$\|Tx^k - x^k\|$$
$$\leq \|Tx^k - Tz^{k+1}\| + \|Tz^{k+1} - x^k\|$$
$$\leq \|x^k - z^{k+1}\| + \|2x^{k+1} - z^{k+1} - x^k\|$$
$$\leq \|x^k - z^{k+1}\| + \|x^{k+1} - z^{k+1}\| + \|x^{k+1} - x^k\|. \quad (18)$$

Combining (17) and (18), we get $\|Tx^k - x^k\| \to 0$. Using $x^{k_i} \to x^*$, we obtain $x^* \in \text{Fix}(T)$.

On the other hand, from (15), for all $t \in \text{Fix}(T)$, we have

$$\|x^{k+1}-t\|^2 - \|x^k - t\|^2 \leq 2\lambda_{k_0}\langle A(y^k), t - y^k\rangle \quad \forall k \geq k_0,$$

or

$$\langle x^k - x^{k+1}, x^k + x^{k+1} - 2t\rangle$$
$$+2\lambda_{k_0}\langle A(y^k), t - y^k\rangle \geq 0 \quad \forall k \geq k_0. \quad (19)$$

In (19), letting $k = k_i$, taking limit as $i \to \infty$, noting that $\|x^k - x^{k+1}\|^2 \to 0, \|x^k - y^k\| \to 0, \{x^k\}$ is bounded and A is continuous, we obtain

$$\langle A(x^*), t - x^*\rangle \geq 0 \quad \forall t \in \text{Fix}(T).$$

Hence, $x^* \in Sol(A, \text{Fix}(T))$. Since the sequence $\{\|x^k - x^*\|\}$ is convergent and $x^{k_i} \to x^*$, we infer that $x^k \to x^*$.

This is end of the proof of Claim 2. So, Theorem 3.2 is proven.

Remark 3.3. Condition (6) was used in many paper on this topic [5, 6]. Obviously, it is satisfied if there exists a number $k_0 \geq 0$ such that $y^k \in \text{Fix}(T)$ for all $k \geq 0$. In the next corollary, we consider a relaxed version of this condition.

Corollary 3.4. *Suppose that Assumptions* (A1)-(A3) *in Assumption 3.1 hold. Moreover, there exists a number* $k_0 \geq 0$ *such that*

$$\Omega := \{ z \in \text{Fix}(T) : \langle Ay^k, z - y^k\rangle \leq 0 \quad \forall k \geq k_0\} \neq \emptyset \quad (20)$$

Then, problem (4) has at least one solution and each cluster point of $\{x^k\}$ *generated by Algorithm 1 is a solution of this problem.*

Proof. The proof of this corollary is inferred directly from Theorem 3.2 and is therefore omitted.

3.2. Second algorithm

In practice, condition (6) and the nonemptiness of the solution set of (4) are difficult to verify. We introduced a modified version of Algorithm 1, in which, the nonemptiness of the solution set and convergence of the algorithm are guaranteed by conditions that are easier to verify.

Algorithm 2.

Step 0. Choose $x^0 \in \mathbb{R}^m; \{\lambda_k\} \in (0,\infty)$ satisfying $\lim_{k\to\infty}\lambda_k = 0$. Set $k = 0$.

Step 1. Given x^k. Compute

$$y^{k+1} = x^k - \lambda_k A(x^k)$$
$$z^{k+1} = x^k - \lambda_k A(y^{k+1})$$
$$x^{k+1} = \tfrac{1}{2}(z^{k+1} + T(z^{k+1})).$$

Step 2. Update $k := k+1$ and GOTO **Step 1.**

Theorem 3.5. *Suppose that Assumptions* (A1)-(A3) *in Assumption 3.1 hold and the sequence* $\{x^k\}$ *generated by Algorithm 2 is bounded. Moreover, assume that*

$$\|x^{k+1} - x^k\| = o(\lambda_k), \quad (21)$$

Then, problem (4) has at least one solution and each cluster point of $\{x^k\}$ *is a solution of this problem.*

Proof. Since $\{x^k\}$ is bounded, using the definitions of y^k and z^k, we infer that the sequences $\{y^k\}$ and $\{z^k\}$ are also bounded. Take $\rho \in (0,1)$ arbitrarily. Since A is Lipschitz continuous and $\lambda_k \to 0$, without loss of generality, we may assume that

$$\lambda_k\|A(x^k) - A(y^{k+1})\| \leq \rho\|x^k - y^{k+1}\| \ \forall k \geq 0.$$

Applying similar arguments that led us to (15), for all $t \in \text{Fix}(T)$, we have

$$\|x^{k+1} - z^{k+1}\|^2 + (1-\rho)\|z^{k+1} + y^{k+1}\|^2$$
$$+(1-\rho)\|y^{k+1} - x^k\|^2$$
$$\leq \langle x^k - x^{k+1}, x^k + x^{k+1} - 2t\rangle$$
$$+2\lambda_k\langle A(y^{k+1}), t - y^{k+1}\rangle. \quad (22)$$

Since $\lambda_k \to 0$, $\|x^{k+1} - x^k\| = o(\lambda_k)$ and $\{y^k\}$ is bounded, the right hand side term of (22) tends to zero. Thus,

$$\lim_{k\to\infty}\|x^{k+1} - z^{k+1}\| = \lim_{k\to\infty}\|z^{k+1} - y^{k+1}\|$$
$$= \lim_{k\to\infty}\|y^{k+1} - x^k\| = 0. \quad (23)$$

Let $\{x^{k_j}\}$ is a subsequence of $\{x^k\}$ satisfying $\lim_{j\to\infty} x^{k_j} = \bar{x}$. We will prove that $\bar{x} \in Sol(A, \text{Fix}(T))$. Using (23) and similar arguments that led us to (18), we have $\bar{x} \in \text{Fix}(T)$. From (22), we have

$$\langle A(y^{k+1}), t - y^{k+1}\rangle$$
$$\geq -\frac{\|x^k - x^{k+1}\|\|x^k + x^{k+1} - 2t\|}{2\lambda_k}. \quad (24)$$

In (24), let $k = k_i$ and take the limit as $j \to \infty$. Noting that $\|x^{k+1} - x^k\| = o(\lambda_k)$ and $\|y^{k+1} - x^k\| \to 0$, we have

$$\langle A(\bar{x}), t - \bar{x}\rangle \geq 0 \ \forall t \in \text{Fix}(T).$$

4. Numerical Experiments

In this section, we present two numerical examples to verify the effectiveness of the proposed algorithms. Also, we compare our algorithms with some existing ones. Numerical experiments were conducted using Matlab version R2014, running on a PC with CPU i3 4150 and 8GB Ram.

Example 1. Let $T \coloneqq 2P_C - I$ where

$$C \coloneqq \{x \in \mathbb{R}^m : Dx \leq b, x_i \geq -1 \ \forall i = 1, \ldots, m\}.$$

$$D = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

I is the identity mapping on $\mathbb{R}^m$, $A : \mathbb{R}^m \to \mathbb{R}^m$, $A(x) = x$ for all $x \in \mathbb{R}^m$. It is easily seen that Assumption 3.1 is satisfied and $x^* = (0,0,0,0,0)^\top$ is the unique solution of (4). From the definition of y^k, we get

$$y^{k+1} = (1 - \lambda_k)x^k.$$

Since $x^k \in \text{Fix}(T)$, it implies that $y^k \in \text{Fix}(T)$ for all $k \geq 1$, and hence, Condition (6) holds.

We compare Algorithm 1 with the Extragradient algorithm (EGD) and the Ergodic algorithm (ERG).

The parameters of these algorithms are chosen as follows:

- In our algorithm, we choose $\rho = \delta = 0.7$, $\lambda_{-1} = 1$;
- In (EGD), since $L = 1$, it implies that $\lambda \in (0,1)$. We have tested this algorithm with $\lambda = 0.1, 0.2, \ldots, 0.9$ and found that the algorithm seems to perform best with $\lambda = 0.5$. We will use this value of the parameter in our comparisons.
- In (ERG), we choose $\lambda_k = \frac{1}{k+1}$ for all $k \geq 0$.

In all the algorithms, we use the same stopping rule $error \leq 10^{-3}$, where $error = \|x^k - x^*\|$ in Algorithm 2, (EGD) and $error = \|z^k - x^*\|$ in (ERG), the same starting point x^0, which is randomly generated. We have tested the algorithms with different m. The results are presented in Table 1, Fig. 1, Fig. 2. (Dash (-) indicates that the computational time of the algorithm is greater than 200 seconds). We can see that our algorithm shows a better behavior in terms of computational time.

Table 1. Performance of the three algorithms in Example 4.1.

	(EGD)		(ERG)		Alg. 1	
	Times[s]	Iter.	Times[s]	Iter.	Times[s]	Iter.
$m = 5$	0.2170	28	-	-	0.1155	34
$m = 50$	0.4398	32	-	-	0.2700	39
$m = 100$	1.7870	34	-	-	1.0327	41
$m = 200$	11.4278	35	-	-	6.7803	42
$m = 500$	163.3687	36	-	-	109.3060	44

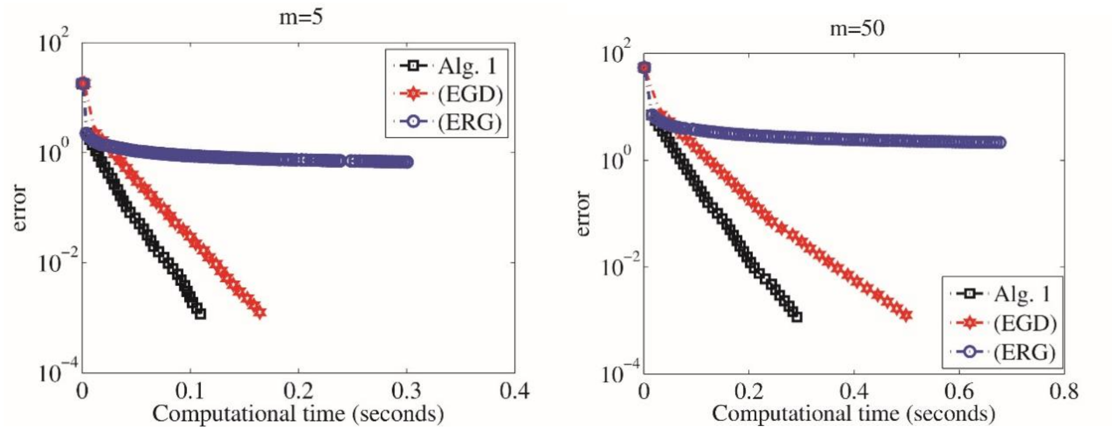

Fig. 1. Performance of the three algorithms in Example 4.1, $m = 5$ and $m = 50$

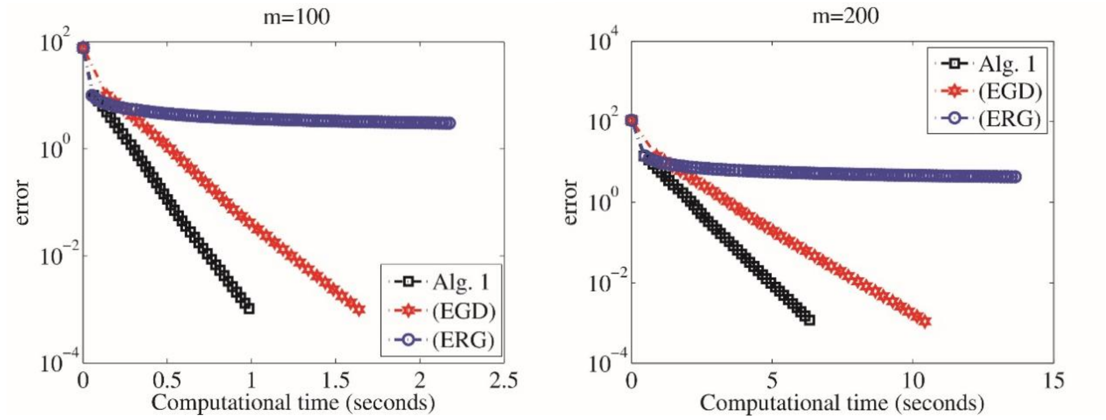

Fig. 2. Performance of the three algorithms in Example 4.1, $m = 100$ and $m = 200$

Example 2. We compare Algorithm 2 with (EGD) and (ERG) in the following problem:

$$T = 2P_C - I,$$

$$C \coloneqq \{x \in \mathbb{R}^m : 2x_1^2 + x_2^2 + \cdots + x_m^2 \leq 1\},$$

$$A : \mathbb{R}^{\mathrm{m}} \to \mathbb{R}^{\mathrm{m}}, A(x) = Bx,$$

where $B = (b_{ij})_{m\times n}$,

$$b_{ij} = \begin{cases} 1 & if\ i = j\ or\ j < i = m - j, \\ -1 & if\ i < j = m - i, \\ 0 & otherwise. \end{cases}$$

It is easily seen that all the convergence conditions of the algorithms are satisfied and $Sol(A, \mathrm{Fix}(T)) = \{x^* = (0, 0, \dots, 0)^\top\}$, $L = 1$. We implement the algorithms with the following parameters:

1. In Algorithm 2, we choose $\lambda_k = \frac{1}{k+1}$;

2. In (EGD), $\lambda = 0.5$;

3. In (ERG), $\lambda_k = \frac{1}{k+1}$ and $T = P_C$.

In Algorithm 2, to verify condition (21), we investigate the sequence $\alpha_k \coloneqq \frac{\|x^{k+1} - x^k\|}{\lambda_k}$. The behavior of this sequence is presented in Fig. 3(b), 4(b), 5(b), 6(b), 7(b). From these figures, we see that $\alpha_k \to 0$, and thus, condition (21) is satisfied. In the three algorithms, we use the same starting point x^0, which is randomly generated and the same stopping rule $error \leq 10^{-3}$, where $error = \|x^k - x^*\|$ in Algorithm 2, (EGD) and $error = \|z^k - x^*\|$ in (ERG). The comparisons are presented in Fig. 3(a), 4(a), 5(a), 6(a), 7(a). We can see that the new algorithm shows a better behavior in terms of computational time.

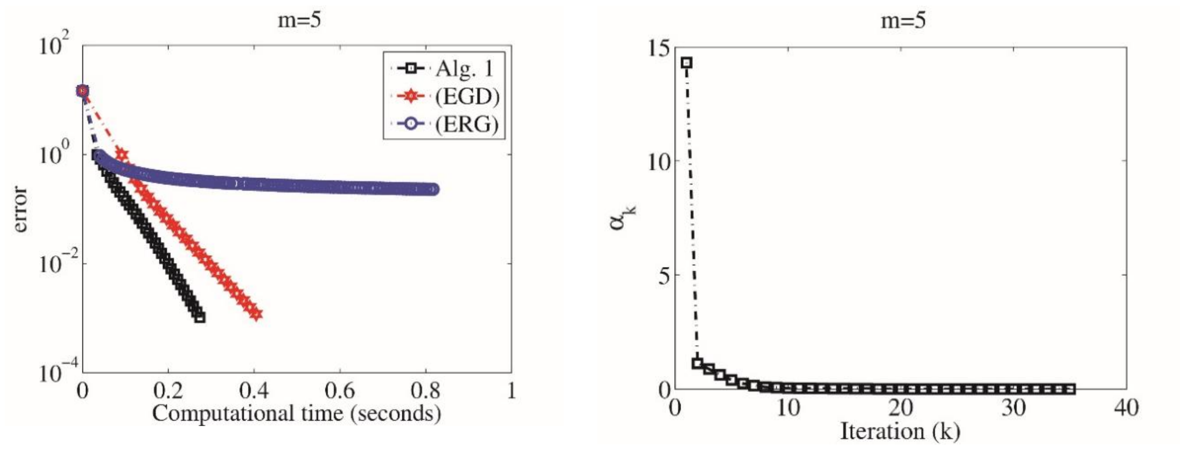

Fig. 3. Performance of the three algorithms in Example 4.2, $m = 5$

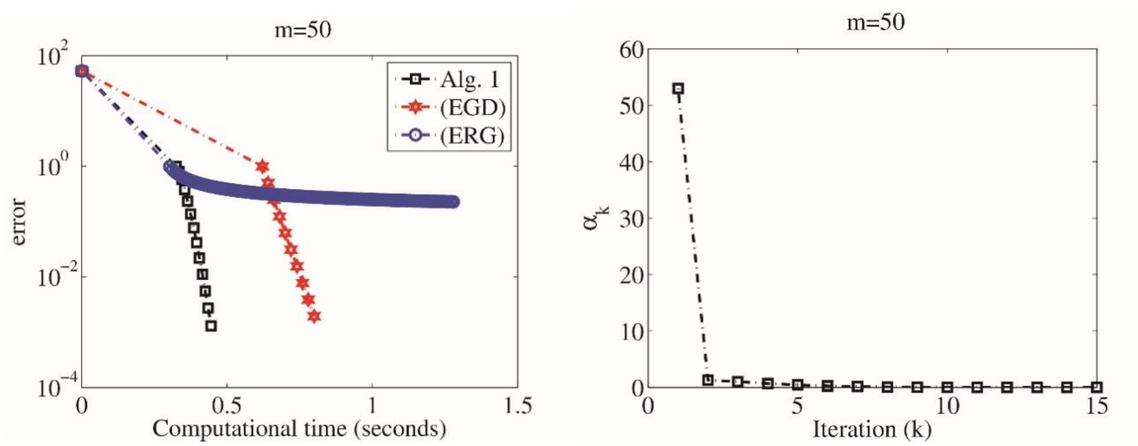

Fig. 4. Performance of the three algorithms in Example 4.2, $m = 5$

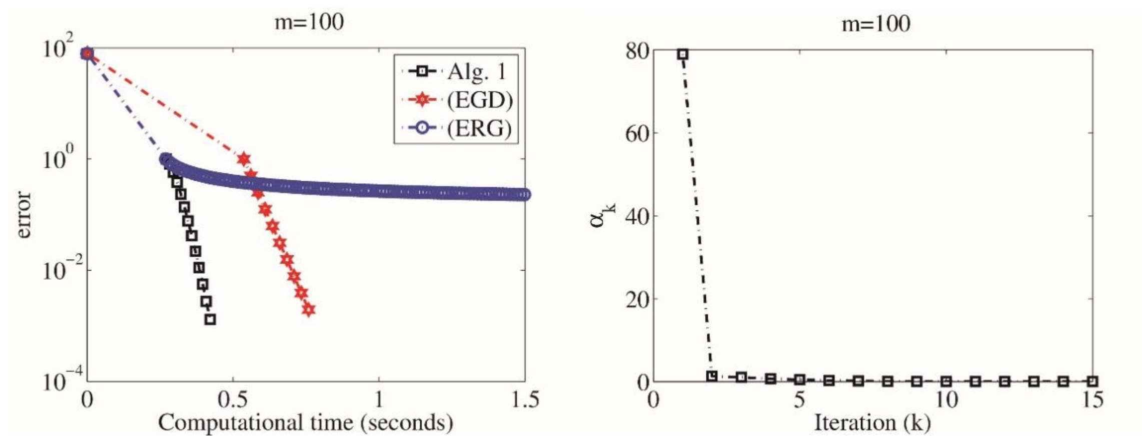

Fig. 5. Performance of the three algorithms in Example 4.2, $m = 100$

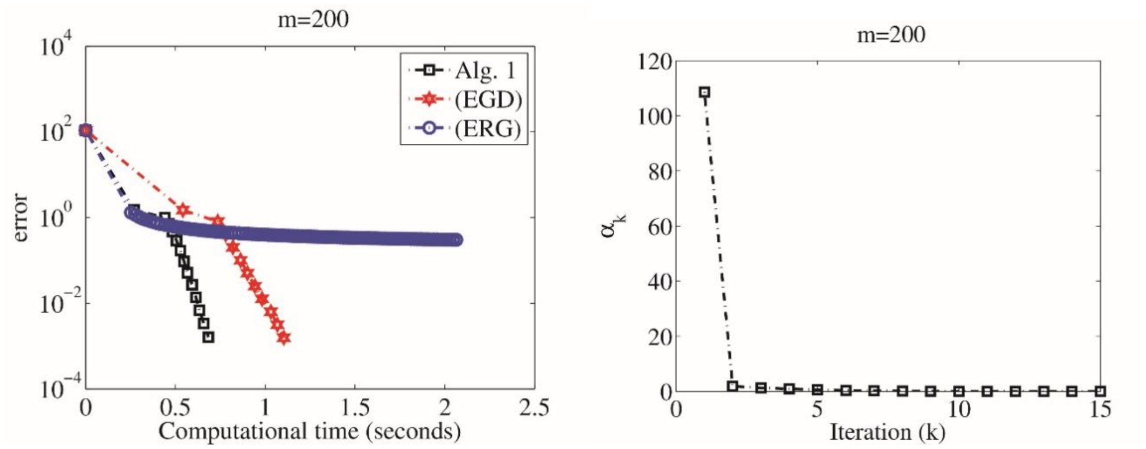

Fig. 6. Performance of the three algorithms in Example 4.2, $m = 200$

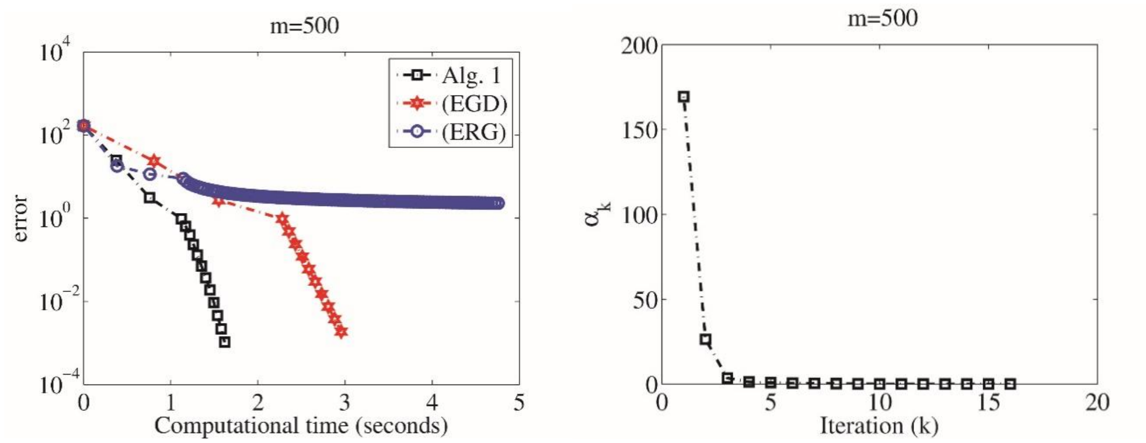

Fig. 7. Performance of the three algorithms in Example 4.2, $m = 500$

5. Conclusion

In this paper, we have introduced two algorithms for solving variational inequalities over the fixed point set of a nonexpansive mapping. The new algorithms can be considered as improved versions of some existing ones. The main advantage of the new algorithms is that they do not require to perform any projection in their steps. This feature helps greatly to reduce the computational cost of the proposed algorithms. Moreover, our algorithm does not require to know the Lipschitz constant of the involving mapping. Also, we have performed some numerical experiments to test the effectiveness of these algorithms.

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