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Boundedness and Stability of Solutions to the Non-autonomous Oseen-Navier-Stokes Equation

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Abstract

We consider the motion of a viscous imcompressible fluid past a rotating rigid body in three-dimensional, where the translational and angular velocities of the body are prescribed but time-dependent. In a reference frame attached to the body, we have the non-autonomous Oseen-Navier-Stokes equations in a fixed exterior domains. We prove the existence and stability of bounded mild solutions in time t to ONSE in three-dimensional exterior domains when the coefficients are time dependent. Our method is based on the $L^p - L^q$-estimates of the evolution family $\big(U(t,s)\big)$ and that of its gradient to prove boundedness of solution to linearized equations. After, we use fixed-point arguments to obtain the result on boundedness of solutions to non-linearized equations when the data belong to L^p-space and are sufficiently small. Finally, we prove existence and polynomial stability of bounded solutions to ONSE with the same condition. Our result is useful for the study of the time-periodic mild solution to the non-autonomous Oseen-Navier-Stokes equations in an exterior domains.

Keywords: boundedness and stability of solutions, exterior domains, non-autonomous equations, Oseen-Navier-Stokes flows.

1. Introduction

The motion of compact obstacles or rigid bodies in a viscous and incompressible fluid is a classical problem in fluid mechanics, and it is still in the focus of applied research. It is interesting to consider the flow of viscous incompressible fluids around a rotating obstacle, where the rotation is prescribed. The rotation of the obstacle causes interesting mathematical problems and difficulties. Moreover, this problem brings out various applications such as applications to windmill, wind energy, as well as airplane designation, and so on. Therefore, this problem has been attracting a lot of attention for the last 20 years. The stability of solutions to Navier-Stokes equations (NSE) can be traced back to Serrin (1959). He proved exponential stability of solutions as well as the existence of time-periodic solutions to NSE in bounded domains.

This direction has been extended further by Miyakawa and Teramoto, Kaniel and Shinbrot (1967), and so on. Maremonti proved the existence and stability of bounded solutions to NSE on the whole space. Kozono and Nakao defined a new notion of mild solutions; their existence on the whole time-line. Then, Taniuchi proved the asymptotic stability of such solutions.

In the present paper, we consider the 3-dimensional Navier-Stokes flow past an obstacle, which is a moving rigid body with prescribed translational and angular velocities. Let Ω is an exterior domain in $\mathbb{R}^3$ with $C^{1,1}$-boundary $\partial\Omega$. Complement $\mathbb{R}^3\backslash\Omega$ is identified with the obstacle (rigid body) immersed in a fluid, and it is assumed to be a compact set in $B(0)$with nonempty interior. After rewriting the problem on a fixed exterior domain $\Omega \in \mathbb{R}^3$, the system is reduced to

$$\begin{cases} u_t + (u.\nabla)u - \Delta u + \nabla p = (\eta + \omega \times x).\nabla u \\ \qquad -\omega \times u + \mathrm{div}F \\ \nabla.u = 0 \\ u|_{\partial\Omega} = \eta + \omega \times x \\ u(.,0) = u_0 \\ u \to 0 \ as \ |x| \to \infty \end{cases} \quad (1)$$

in $\Omega \times (0,\infty)$, where $\{u(x,t), p(x,t)\}$ with $u = (u_1, u_2, u_3)^T$ is the pair of unknowns which are the velocity vector field and pressure of a viscous fluid, respectively, while the external force $\mathrm{div}F$ being a second-order tensor field. Meanwhile, $\eta(0,0,a(t))^T$ and $\omega = (0,0,k(t))^T$stand for the translational and angular velocities respectively of the obstacle. Here and in what follows, $(.)^T$ stands for the transpose of vectors or matirices. Such a time-dependent problem was first studied by Borchers [1] in the framework of weak solutions. The result has then been extended further by many authors, e.g., Hishida [2, 3],

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Galdi [4, 5]. Hansel and Rhandi [6, 7] succeeded in the proof of generation of this evolution operator with the $L^p - L^q$ smoothing rate. They constructed evolution operator in their own way since the corresponding semigroup is not analytic (Hishida [2]). Recently, Hishida [3] developed the $L^p - L^q$ decay estimates of the evolution operator see Proposition 1.2. However, it is difficult to perform analysis with the standard Lebesgue space on account of the scale-critical pointwise estimates. Thus, we first construct a solution for the weak formulation in the framework of Lorentz space by the strategy due to Yamazaki [8]. We next identify this solution with a local solution possessing better regularity in a neighborhood of each time. Moreover, Huy [9] showed that the existence and stability of bounded mild periodic solutions to the NSE passing an obstacle which is rotating around certain axes .

Our conditions on the translational and angular velocities are

$$\eta, \omega \in C^\theta([0,\infty);\ \mathbb{R}^3) \cap C^1([0,\infty);\ \mathbb{R}^3) \cap L^\infty(0,\infty;\ \mathbb{R}^3) \text{ with some } \theta \in (0,1). \tag{2}$$

Lets us introduce the following notations:

$$|(\eta,\omega)|_0 := \sup_{t\geq 0}(|\eta(t)| + |\omega(t)|),$$

$$|(\eta,\omega)|_1 := \sup_{t\geq 0}(|\eta'(t)| + |\omega'(t)|),$$

$$|(\eta,\omega)|_\theta := \sup_{t>s\geq 0} \frac{|\eta(t)-\eta(s)| + |\omega(t)-\omega(s)|}{(t-s)^\theta}.$$

There is a constant $m \in (0,\infty)$ such that

$$|(\eta,\omega)|_0 + |(\eta,\omega)|_1 + |(\eta,\omega)|_\theta \leq m \tag{3}$$

Let us begin with introducing notation. Given an exterior domain Ω of class $C^{1,1}$ in $\mathbb{R}^3$, we consider the following spaces:

$$C_{0,\sigma}^\infty(\Omega) := \{v \in C_0^\infty(\Omega) : \nabla . v = 0 \text{ in } \Omega\},$$

$$L_\sigma^p(\Omega) := \overline{C_{0,\sigma}^\infty(\Omega)}^{\|\cdot\|_{L^p}} .$$

we also need the notion of Lorentz space $L^{r,q}(\Omega), (1 < r < \infty, 1 \leq q \leq \infty)$ is defined by

$L^{r,q}(\Omega) := \{f$: Lebesgue measurable function $|\ \|f\|^*_{r,q} < \infty\}$

where

$$\|f\|^*_{r,q} = \begin{cases} \left(\int_0^\infty \frac{\left(t\mu(\{x\in\Omega | f(x) > t\})^{\frac{1}{q}}\right)^r dt}{t}\right)^{\frac{1}{r}} & 1 \leq r < \infty \\ \sup_{t>0} t\mu(\{x\in\Omega | f(x) > t\})^{\frac{1}{q}} & r = \infty \end{cases}$$

and $\mu(.)$ denotes the Lebesgue measure on $\mathbb{R}^3$. The spaces $L^{r,q}(\Omega)$ is a quasi−normed space and it is even a Banach space equipped with norm $\|.\|_{r,q}$ equivalent to $\|.\|^*_{r,q}$ and note that $L^{r,r}(\Omega) = L^r(\Omega)$ and that for $q = \infty$ the space $L^{r,\infty}(\Omega)$ is called the weak L^r −space and is denoted by $L^r_w(\Omega) := L^{r,\infty}(\Omega)$. We denote various constants by C and they may change from line to line. The constant dependent on $A, B, \cdots$ is denoted by $C(A, B, \ldots)$. Finally, if there is no confusion, we use the same symbols for denoting spaces of scalar-valued functions and those of vector-valued ones.

The following weak Holder inequality is known (see [10, Lemma 2.1]):

Lemma 1.1.

Let $1 < p \leq \infty, 1 < q < \infty$ and $1 < r < \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. If $f \in L^p_w, g \in L^q_w$ then $fg \in L^r_w$ and

$$\|fg\|_{r,w} \leq C\|f\|_{p,w}\|g\|_{q,w} \tag{4}$$

where C is a positive constant depending only on p and q. Note that $L^\infty_w = L^\infty$.

Let $\mathbb{P} = \mathbb{P}_r$ be the Helmholtz projection on $L^r(\Omega)$. Then, $\mathbb{P}$ defines a bounded projection on each $L^{r,q}(\Omega), (1 < r < \infty, 1 \leq q \leq \infty)$ which is also denoted by $\mathbb{P}$. We have the following notations of solenoidal Lorentz spaces:

$$L^{r,q}_\sigma(\Omega) := \mathbb{P}\big(L^{r,q}(\Omega)\big)$$

Then we can see that

$$L^{r,q}(\Omega) = L^{r,q}_\sigma(\Omega) \oplus \{\nabla p \in L^{r,q} : p \in L^{r,q}_{loc}(\overline{\Omega})\}$$

We also have

$$L^{r,q}_\sigma(\Omega) := \left(L^{r_1}_\sigma(\Omega), L^{r_2}_\sigma(\Omega)\right)_{\theta,q}$$

where

$$1 < r_1 < r < r_2 < \infty, 1 \leq q \leq \infty, \frac{1}{r} = \frac{1-\theta}{r_1} + \frac{\theta}{r_2}$$

and $(.,.)_{\theta,q}$ denotes the real interpolation functor. Furthermore, if $1 \leq q < \infty$ then

$\left(L^{r,q}_\sigma\right)' = L^{r',q'}_\sigma$ here $r' = \frac{r}{r-1}, q' = \frac{q}{q-1}$ and $q' = \infty$ if $q = 1$.

When $q = \infty$ let $L^s_{\sigma,w}(\Omega) = L^{s,\infty}_\sigma(\Omega)$ and write $\|.\|_{s,w}$ for the norm in $L^s_{\sigma,w}(\Omega)$. We also need the following space of bounded continuous functions on $\mathbb{R}_+ := (0,\infty)$ with values in $L^s_{\sigma,w}(\Omega)$:

$$C_b\left(\mathbb{R}_+, L^s_{\sigma,w}(\Omega)\right) := \left\{v : \mathbb{R}_+ \to L^s_{\sigma,w}(\Omega) \mid v \text{ is continuous and } \sup_{t\in\mathbb{R}_+} \|v(t)\|_{s,w} < \infty\right\}$$

endowed with the norm

$$\|v\|_{\infty,s,w} := \sup_{t\in\mathbb{R}_+} \|v(t)\|_{s,w}.$$

Next, for each $t \geq 0$ we consider the operator $L(t)$ as follows:

$$D(\mathcal{L}(t)) := \begin{Bmatrix} u \in L^{\,r}_{\sigma} \cap W^{1,r}_0 \cap W^{2,r}: \\ (\omega(t)\times x).\nabla u \in L^r(\Omega) \end{Bmatrix}$$

$$\mathcal{L}(t)u := \mathbb{P}[\Delta u + (\eta + \omega \times x).\nabla u - \omega \times u] \qquad (5)$$

for $u \in D(\mathcal{L}(t))$.

It is known that the family of operators $\{\mathcal{L}(t)\}_{t\geq 0}$ generates a bounded evolution family $\{U(t,s)\}_{t\geq s\geq 0}$ on $L^{\,r}_{\sigma}(\Omega))$ for each $1 < r < \infty$ under the conditions (2). Then $\{U(t,s)\}_{t\geq s\geq 0}$ is extended to a strongly continuous, bounded evolution operator on $L^{\,r,q}_{\sigma}(\Omega)$.

We recall the following $L^{r,q} - L^{p,q}$ estimates taken from [4].

Proposition 1.2.

Suppose that η and ω fulfill (2) and (3) for each $m \in (0,\infty)$.

(i) Let $1 < p \leq r < \infty, 1 \leq q \leq \infty$, there is a constant $C = C(m,p,q,r,\theta,\Omega)$ such that

$$\|U(t,s)x\|_{r,q}, \|U(t,s)^*x\|_{r,q} \leq C(t-s)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{r}\right)}\|x\|_{p,q} \qquad (6)$$

for all $t > s \geq 0$.

(ii) Let $1 < p \leq r < 3, 1 \leq q \leq \infty$, there is a constant $C = C(m,p,q,r,\theta,\Omega)$ such that

$$\|\nabla U(t,s)x\|_{r,q} \leq C(t-s)^{-\frac{1}{2}-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{r}\right)}\|x\|_{p,q} \qquad (7)$$

for all $t > s \geq 0$.

(iii) When $1 < p \leq r \leq 3, 1 \leq q \leq \infty$, there is a constant $C = C(m,p,q,r,\theta,\Omega)$ such that

$$\|\nabla U(t,s)^*x\|_{r,q} \leq C(t-s)^{-\frac{1}{2}-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{r}\right)}\|x\|_{p,q} \qquad (8)$$

for all $t > s \geq 0$.

If in particular $\frac{1}{p} - \frac{1}{r} = \frac{1}{3}$ as well as $1 < p \leq r \leq 3$, there is a constant $C = C(m,p,r,\theta,\Omega)$ such that

$$\int_0^t \|\nabla U(t,s)^*x\|_{r,1}\, ds \leq C\|x\|_{p,1} \qquad (9)$$

for all $t > s \geq 0$.

Proof. We use the interpolation theorem and $L^p - L^q$ decay estimates in Hishida [3] we obtain the estimate (6) and (7). The assertions (iii) have been proved in [4].

We fix a cut-off function $\phi \in C_0^\infty(B_{3R_0})$ such that $\phi = 1$ on B_{2R_0}, where R_0 satisfy

$$\mathbb{R}^3\backslash\Omega \subset B_{R_0} := \{x \in \mathbb{R}^3; |x| < R_0\}.$$

We define

$$b(x,t) = \frac{1}{2}\operatorname{rot}\{\phi(\eta \times x - |x|^2\,\omega)\} \qquad (10)$$

which fulfills

$$\operatorname{div} b = 0, b|_{\partial\Omega} = \eta + \omega \times x, b(t) \in C_0^\infty(B_{3R_0})$$

By straightforward computations, we have $\omega \times b = \operatorname{div}(-F_1)$, $b_t = \operatorname{div}(-F_2)$ for

$$F_1 = \begin{pmatrix} \frac{(a(t))^2|x|^2\phi(x)}{2} & 0 & -a(t)k(t)x_2\phi(x) \\ 0 & \frac{(a(t))^2|x|^2\phi(x)}{2} & a(t)k(t)x_1\phi(x) \\ 0 & 0 & 0 \end{pmatrix}$$

$$F_2 = \begin{pmatrix} 0 & \frac{a\prime(t)|x|^2\phi(x)}{2} & \frac{k\prime(t)x_1\phi(x)}{2} \\ \frac{-a\prime(t)|x|^2\phi(x)}{2} & 0 & \frac{k\prime(t)x_2\phi(x)}{2} \\ -k'(t)x_1\phi(x) & -k'(t)x_2\phi(x) & 0 \end{pmatrix}$$

By setting $u := z + b$ problem (1) is equivalent to

$$\begin{cases} \left.\begin{matrix} z_t - \Delta z - (\eta + \omega \times x).\nabla u + \omega \times z + \nabla p \\ +(z.\nabla)z + (z.\nabla)b + (p.\nabla)z + (b.\nabla)b \end{matrix}\right\} = \operatorname{div} G \\ \nabla . z = 0 \\ z|_{\partial\Omega} = 0 \\ z(.,0) = z_0 \\ z \to 0 \text{ as } |x| \to \infty \end{cases} \qquad (11)$$

where $z_0(x) = u_0(x) - b(x,0)$

and

$$G = F + F_1 + F_2 + \Delta b + (\eta + \omega \times x)\otimes\nabla b. \qquad (12)$$

Applying Helmholtz operator $\mathbb{P}$ to (1) we may rewrite the equation as a non-autonomous abstract Cauchy problem.

$$\begin{cases} z_t + \mathcal{L}(t)z = \mathbb{P}\operatorname{div}(G - z\otimes z - z\otimes b - b\otimes z - b\otimes b) \\ z|_{t=0} = z_0 \end{cases} \qquad (13)$$

where $\mathcal{L}(t)$ is defined as in (5).

2. Bounded Solutions

2.1. The Linearized Problem

In this subsection we study the linearized non-autonomous system associated to (13) for some initial value $z_0 \in L^{\,3}_{\sigma,w}(\Omega)$.

$$\begin{cases} z_t + \mathcal{L}(t)z = \mathbb{P}\operatorname{div}(G) \\ z|_{t=0} = z_0 \end{cases} \qquad (14)$$

We can define a *mild solution* of (14) as the function $z(t)$ fulfilling the following integral equation in which the integral is understood in weak sense as in [11]

$$z(t) = U(t,0)z(0) + \int_0^t U(t,\tau)\,\mathbb{P}\mathrm{div}\big(G(\tau)\big)d\tau. \tag{15}$$

Remark 2.1.

Let η and ω satisfy both (2) and (3). Let the external force $F \in C_b\left(\mathbb{R}_+, L^{\frac{3}{2}}_{\sigma,w}(\Omega)^{3\times3}\right)$

Then G belongs to $C_b\left(\mathbb{R}_+, L^{\frac{3}{2}}_{\sigma,w}(\Omega)^{3\times3}\right)$, moreover

$$\|G\|_{\infty,\frac{3}{2},w} \le \|F\|_{\infty,\frac{3}{2},w} + Cm + C'm^2. \tag{16}$$

The following theorem contains our first result on the boundedness of mild solutions of the linear problem.

Theorem 2.2.

Suppose that η and ω fulfill both (2) and (3), the external force F belongs to $C_b\left(\mathbb{R}_+, L^{\frac{3}{2}}_{\sigma,w}(\Omega)^{3\times3}\right)$ and let $z_0 \in L^{3}_{\sigma,w}(\Omega)$.

Then, problem (14) has a unique mild solution $z \in C_b\big(\mathbb{R}_+, L^{3}_{\sigma,w}(\Omega)^{3\times3}\big)$ expressed by (15) with $z(0) = z_0$. Moreover, we have

$$\|z\|_{\infty,3,w} \le C'^{\|z_0\|_{3,w}} + \hat{C}\|G\|_{\infty,\frac{3}{2},w} \tag{17}$$

where $C', \hat{C}$ are certain positive constants independent of z_0, z, and G.

Proof. Firstly, for $z_0 \in L^{3}_{\sigma,w}(\Omega)$, we prove that the function z defined by (15) belong to $C_b\big(\mathbb{R}_+, L^{3}_{\sigma,w}(\Omega)^{3\times3}\big)$

Indeed, for each $\varphi \in L^{\frac{3}{2},1}_{\sigma}$ we estimate

$$\begin{aligned}
&|\langle z(t),\varphi\rangle| \\
&\le |\langle U(t,0)z_0,\varphi\rangle| + \left|\left\langle \textstyle\int_0^t U(t,\tau)\mathbb{P}\mathrm{div}G(\tau)d\tau,\varphi\right\rangle\right| \\
&\le |\langle U(t,0)z_0,\varphi\rangle| + \int_0^t |\langle U(t,\tau)\mathbb{P}\mathrm{div}G(\tau),\varphi\rangle| d\tau \\
&\le |\langle U(t,0)z_0,\varphi\rangle| + \textstyle\int_0^t |\langle G(\tau),\nabla U(t,\tau)^*\varphi\rangle| d\tau \\
&\le \|U(t,0)z_0\|_{3,w}\|\varphi\|_{\frac{3}{2},1} \\
&\qquad + \int_0^t \|G(\tau)\|_{\frac{3}{2},w}\|\nabla U(t,\tau)^*\varphi\|_{3,1}d\tau \\
&\le C'\|z_0\|_{3,w}\|\varphi\|_{\frac{3}{2},1} \\
&+ \|G\|_{\infty,\frac{3}{2},w}\int_0^t \|\nabla U(t,\tau)^*\varphi\|_{3,1}d\tau.
\end{aligned} \tag{18}$$

We now use the $L^{r,q} - L^{p,q}$ smoothing properties (see Prop. 1.2) yielding that $\int_0^t \|\nabla U(t,s)^*\varphi\|_{3,1}\,ds \le \hat{C}\|\varphi\|_{\frac{3}{2},1}$.

Plugging this inequality to (18) we obtain $|\langle z(t),\varphi\rangle| \le C'\|z_0\|_{3,w}\|\varphi\|_{\frac{3}{2},1} + \hat{C}\|G\|_{\infty,\frac{3}{2},w}\|\varphi\|_{\frac{3}{2},1}$ for all $t > 0$ and all $\varphi \in L^{\frac{3}{2},1}_{\sigma}$.

This implies that

$$\|z(t)\|_{3,w} \le C'\|z_0\|_{3,w} + \hat{C}\|G\|_{\infty,\frac{3}{2},w}\ \forall\, t \ge 0. \tag{19}$$

Let us show the weak-continuity of $z(t)$ with respect to $t \in (0,\infty)$ with values in $L^{3}_{\sigma,w}$. Since, $U(t,s)$ is strongly continuous, we have that $U(t,0)z_0$ is continuous w.r.t to t. Therefore, we only have to prove that the integral function $\int_0^t U(t,\tau)\,\mathbb{P}\mathrm{div}\big(G(\tau)\big)d\tau$ is continuous w.r.t to t. To this purpose, for $\varphi \in C^\infty_{0,\sigma}(\Omega)$ ($C^\infty_{0,\sigma}(\Omega)$is dense in $L^{\frac{3}{2},1}_{\sigma}$). It is sufficient to show that

$$\left|\left\langle\left(\int_0^t U(t,\tau)\,\mathbb{P}\mathrm{div}\big(G(\tau)\big)d\tau - \int_0^s U(t,\tau)\,\mathbb{P}\mathrm{div}\big(G(\tau)\big)d\tau\right),\varphi\right\rangle\right| \to 0\ as\ t \to s$$

We suppose $t \ge s \ge \tau$, we estimate

$$\begin{aligned}
&\left|\left\langle \textstyle\int_0^t U(t,\tau)\mathbb{P}\mathrm{div}G(\tau)d\tau - \int_0^s U(s,\tau)\mathbb{P}\mathrm{div}G(\tau)d\tau,\varphi\right\rangle\right| \\
&\le \left|\left\langle \textstyle\int_s^t U(t,\tau)\mathbb{P}\mathrm{div}G(\tau)d\tau,\varphi\right\rangle\right| + \\
&\left|\left\langle \textstyle\int_0^s U(t,\tau)\mathbb{P}\mathrm{div}G(\tau)d\tau - \int_0^s U(s,\tau)\mathbb{P}\mathrm{div}G(\tau)d\tau,\varphi\right\rangle\right| \\
&= \left|\left\langle \textstyle\int_s^t U(t,\tau)\mathbb{P}\mathrm{div}G(\tau)d\tau,\varphi\right\rangle\right| + \\
&\left|\left\langle \int_0^s (U(t,s)-I)U(s,\tau)\mathbb{P}\mathrm{div}G(\tau)d\tau,\varphi\right\rangle\right| = I_1 + I_2
\end{aligned} \tag{20}$$

The first integral can be estimated as

$$\begin{aligned}
I_1 &\le \textstyle\int_s^t |\langle G(\tau),\nabla U(t,\tau)^*\varphi\rangle| d\tau \\
&\le \textstyle\int_s^t \|G\|_{\frac{3}{2},w}\|\nabla U(t,\tau)^*\varphi\|_{3,1}d\tau \\
&\le \|G\|_{\infty,\frac{3}{2},w}\textstyle\int_s^t \|\nabla U(t,\tau)^*\varphi\|_{3,1}d\tau \\
&\le 2C\|G\|_{\infty,\frac{3}{2},w}(t-s)^{\frac{1}{2}}\|\varphi\|_{3,1} \to 0 \text{ as } t \to s.
\end{aligned}$$

Similarly, the second integral I_2 can be estimated by

$$\begin{aligned}
I_2 &\le \int_0^s |\langle G(\tau),\nabla U(s,\tau)^*(U(t,s)^*\varphi-\varphi)\rangle| d\tau \\
&\int_0^s \|G\|_{\frac{3}{2},w}\|\nabla U(t,\tau)^*(U(t,s)^*\varphi-\varphi)\|_{3,1}d\tau \\
&\le \|G\|_{\infty,\frac{3}{2},w}\int_0^s \|\nabla U(t,\tau)^*(U(t,s)^*\varphi-\varphi)\|_{3,1}d\tau \\
&\le C\|G\|_{\infty,\frac{3}{2},w}\|U(t,s)^*\varphi-\varphi\|_{\frac{3}{2},1} \to 0 \text{ as } t \to s.
\end{aligned}$$

We can discuss the other case $s > t > \tau$ similarly

Therefore, the function $z(t)$ is continuous w.r.t. t and we obtain that that $z \in C_b\big(\mathbb{R}_+, L^{\,3}_{\sigma,w}(\Omega)^{3\times 3}\big)$.

2.2. The Nonlinear Problem

In this subsection, we investigate boundedness mild solutions to Oseen-Navier-Stokes equations (13). To do this, similarly to the case of linear equation, we define *the mild solution* to (13) as a function $z(t)$ fulfilling the integral equation

$$z(t) = U(t,0)z(0) + \int_0^t U(t,\tau)\,\mathbb{P}\mathrm{div}(G - z\otimes z - z\otimes b - b\otimes z - b\otimes b)d\tau. \quad (21)$$

The next theorem contains our second main result on the boundedness of mild solutions to nonautonomous Oseen-Navier-Stokes flows.

Theorem 2.3.

Under the same conditions as in theorem 2.2. Then, if $m, \|z_0\|_{3,w}$, $\|F\|_{\infty,\frac{3}{2},w}$ and ρ are small enough, the problem (13) has a unique mild solution $\hat{z}$ in the ball

$$B_\rho := \{v \in C_b\left(\mathbb{R}_+, L^{\,3}_{\sigma,w}(\Omega)\right): \|v\|_{\infty,3,w} \le \rho\}.$$

Proof. We will use the fixed-point arguments. we define the transformation Φ as follows: For $v \in B_\rho$ we set $\Phi(v) = z$ where $z \in C_b\left(\mathbb{R}_+, L^{\,3}_{\sigma,w}(\Omega)\right)$ is given by

$$z(t) = U(t,0)z(0) + \int_0^t U(t,\tau)\,\mathbb{P}\mathrm{div}(G - v\otimes v - v\otimes b - b\otimes v - b\otimes b)d\tau.$$

Next, applying (17) for $G - v\otimes v - v\otimes b - b\otimes v - b\otimes b$ instead of G we obtain

$$\|z\|_{\infty,3,w} \le C'\|z_0\|_{3,w} + \hat{C}\|G - v\otimes v - v\otimes b - b\otimes v - b\otimes b\|_{\infty,\frac{3}{2},w}$$

$$\le C'\|z_0\|_{3,w} + \hat{C}\left(\|G\|_{\infty,\frac{3}{2},w} + \|v\otimes v\|_{\infty,\frac{3}{2},w} + \|v\otimes b\|_{\infty,\frac{3}{2},w} + \|b\otimes v\|_{\infty,\frac{3}{2},w} + \|b\otimes b\|_{\infty,\frac{3}{2},w}\right)$$

$$\le C'\|z_0\|_{3,w} + \hat{C}\left(\|F\|_{\infty,\frac{3}{2},w} + Cm + C'm^2 + C\|v\|^2_{\infty,\frac{3}{2},w} + 2C\|v\|_{\infty,\frac{3}{2},w}\|b\|_{\infty,\frac{3}{2},w} + C\|b\|^2_{\infty,\frac{3}{2},w}\right)$$

$$\le C'\|z_0\|_{3,w} + \hat{C}\left(\|F\|_{\infty,\frac{3}{2},w} + Cm + C'm^2 + C\rho^2 + 2Cm\rho + C\rho^2\right). \quad (22)$$

Thus, for sufficiently small $m, \|z_0\|_{3,w}$, $\|F\|_{\infty,\frac{3}{2},w}$ and ρ, the transformation Φ acts from B_ρ into itself. Moreover, the map Φ can be expressed as

$$\Phi(v)(t) = U(t,0)z(0) + \int_0^t U(t,\tau)\,\mathbb{P}\mathrm{div}(G - v\otimes v - v\otimes b - b\otimes v - b\otimes b)d\tau. \quad (23)$$

Therefore, for $v_1, v_2 \in B_\rho$ we obtain that the difference $\Phi(v_1) - \Phi(v_2)$

$$\big(\Phi(v_1) - \Phi(v_2)\big)(t) = \int_0^t U(t,\tau)\,\mathbb{P}\mathrm{div}(-v_1\otimes v_1 + v_2\otimes v_2 - v_1\otimes b - b\otimes v_1 + v_2\otimes b + b\otimes v_2)d\tau.$$

Applying again (22) we arrive at

$$\|\Phi(v_1) - \Phi(v_2)\|_{\infty,3,w} \le \hat{C}\|-v_1\otimes v_1 + v_2\otimes v_2 - v_1\otimes b - b\otimes v_1 + v_2\otimes b + b\otimes v_2\|_{\infty,\frac{3}{2},w} \le \hat{C}\|-(v_1 - v_2)\otimes v_1 - v_2\otimes(v_1 - v_2) - (v_1 - v_2)\otimes b - b\otimes(v_1 - v_2)\|_{\infty,\frac{3}{2},w} \le \hat{C}(2C\rho + 2Cm)\|v_1 - v_2\|_{\infty,3,w}. \quad (24)$$

Hence, if m and ρ are sufficiently small the map Φ is a contraction. Then, there exists a unique fixed poin $\hat{z}$ of Φ. By definition of Φ, the function $\hat{z}$ is the unique mild solution to (13) and the proof is complete.

3. Stability Solutions

In this section, we consider stability mild solutions to Oseen-Navier-Stokes equations (13).

We then show the polynomial stability of the bounded solutions to (13) in the following theorem.

Theorem 3.1.

Under the same conditions as in theorem 2.2. Then, the small solution $\hat{z}$ of (13) is stable in the sense that for any other solution $u \in C_b\left(\mathbb{R}_+, L^{\,3}_{\sigma,w}(\Omega)\right)$ of (13) such that $\|u(0) - \hat{z}(0)\|_{3,w}$, is small enough, we have

$$\|u(t) - \hat{z}(t)\|_{r,w} \le \frac{C}{t^{\left(\frac{1}{2}-\frac{3}{2r}\right)}} \text{ for all } t > 0 \quad (25)$$

for r being any fixed real number in $(3, \infty)$.

Proof. Putting $v = u - \hat{z}$ we obtain that v satisfies the equation

$$v(t) = U(t,0)(u(0) - \hat{z}(0)) + \int_0^t U(t,\tau)\,\mathbb{P}\mathrm{div}\big(H(v)\big)d\tau \quad (26)$$

where

$$H(v) = -v\otimes(v + \hat{z}) - \hat{z}\otimes v - b\otimes v - v\otimes b. \quad (27)$$

Fix any $r > 3$, set

$$\mathbb{M} = \left\{ v \in C_b\left(\mathbb{R}_+, L^{\,3}_{\sigma,w}(\Omega)\right): \sup_{t>0} t^{\left(\frac{1}{2}-\frac{3}{2r}\right)}\|v(t)\|_{r,w} < \infty\right\} \quad (28)$$

and consider the norm

$$\|v\|_{\mathbb{M}} = \|v\|_{\infty,3,w} + \sup_{t>0} t^{\left(\frac{1}{2}-\frac{3}{2r}\right)}\|v(t)\|_{r,w}. \quad (29)$$

We next clarify that for sufficiently small $m, \|u(0) - \hat{z}(0)\|_{3,w}$ and $\|\hat{z}\|_{\infty,3,w}$, Eq (13) has only one solution in a certain ball of $\mathbb{M}$ centered at 0.

Indeed, for $v \in \mathbb{M}$ we consider the mapping Φ defined formally by

$$\Phi(v)(t) := U(t,0)(u(0) - \hat{z}(0)) + \int_0^t U(t,\tau)\,\mathbb{P}\mathrm{div}\big(H(v)\big)d\tau \quad (30)$$

Denote by $\mathcal{B}_\rho := \{w \in \mathbb{M}: \|w\|_{\mathbb{M}} \le \rho\}$. We then prove that if $m, \|u(0) - \hat{z}(0)\|_{3,w}$ and $\|\hat{z}\|_{\infty,3,w}$ are small enough, the transformation Φ acts from $\mathcal{B}_\rho$ to itself and is a contraction. To this purpose, for $v \in \mathbb{M}$ by a similar way as in the proof of theorem 2.3 we obtain $\Phi(v) \in C_b\left(\mathbb{R}_+, L^{\,3}_{\sigma,w}(\Omega)\right)$. Next, we have

$$t^{\left(\frac{1}{2}-\frac{3}{2r}\right)}\Phi(v)(t) := t^{\left(\frac{1}{2}-\frac{3}{2r}\right)}U(t,0)\big(u(0) - \hat{z}(0)\big) + t^{\left(\frac{1}{2}-\frac{3}{2r}\right)}\int_0^t U(t,\tau)\,\mathbb{P}\mathrm{div}\big(H(v)\big)d\tau$$

By $L^{r,\infty} - L^{3,\infty}$ estimates for evolution operator $U(t,0)$ (see (6)) we derive

$$\left\|t^{\left(\frac{1}{2}-\frac{3}{2r}\right)}U(t,0)\big(u(0) - \hat{z}(0)\big)\right\|_{r,w} \le C\|u(0) - \hat{z}(0)\|_{3,w}.$$

$U(t,s)$ is bounded family

$$\big\|U(t,0)\big(u(0) - \hat{z}(0)\big)\big\|_{3,w} \le C\|u(0) - \hat{z}(0)\|_{3,w}$$

Thus,

$\big\|U(t,0)\big(u(0) - \hat{z}(0)\big)\big\|_{\infty,3,w}$

$\le C\|u(0) - \hat{z}(0)\|_{3,w}$. So, we have

$$\big\|U(t,0)\big(u(0) - \hat{z}(0)\big)\big\|_{\mathbb{M}} = \big\|U(t,0)\big(u(0) - \hat{z}(0)\big)\big\|_{\infty,3,w} + \sup_{t>0} t^{\left(\frac{1}{2}-\frac{3}{2r}\right)}\big\|U(t,0)\big(u(0) - \hat{z}(0)\big)\big\|_{r,w} \le C\|u(0) - \hat{z}(0)\|_{3,w} \quad (31)$$

We consider

$\int_0^t U(t,\tau)\,\mathbb{P}\mathrm{div}\big(H(v)\big)d\tau = \int_0^t U(t,t-\xi)\,\mathbb{P}\mathrm{div}(H(v)(t-\xi))d\xi,\ t > 0$, and estimate this integral. To do this, for any test function $\varphi \in C^\infty_{0,\sigma}(\Omega)$, we have

$$\left|\left\langle \int_0^t U(t,t-\xi)\,\mathbb{P}\mathrm{div}\big(H(v)(t-\xi)\big)d\xi, \varphi\right\rangle\right| = \left|\int_0^t \langle -H(v)(t-\xi), \nabla U(t,t-\xi)^*\varphi\rangle d\xi\right| \le \int_0^t |\langle -H(v)(t-\xi), \nabla U(t,t-\xi)^*\varphi\rangle| d\xi$$
$$= \int_0^{\frac{t}{2}} |\langle -H(v)(t-\xi), \nabla U(t,t-\xi)^*\varphi\rangle| d\xi + \int_{t/2}^t |\langle -H(v)(t-\xi), \nabla U(t,t-\xi)^*\varphi\rangle| d\xi \quad (32)$$

Now, consider the two integrals on the last estimate of (32).

Applying (4) we have

$$\|v\otimes(v+\hat{z})\|_{\frac{3r}{3+r},w} \le C\|v\|_{r,w}\|v+\hat{z}\|_{3,w} \le C\|v\|_{r,w}\big(\|v\|_{3,w} + \|\hat{z}\|_{3,w}\big),$$

$$\|\hat{z}\otimes v\|_{\frac{3r}{3+r},w} \le C\|v\|_{r,w}\|\hat{z}\|_{3,w},$$

$$\|v\otimes b\|_{\frac{3r}{3+r},w} \le C\|v\|_{r,w}\|b\|_{3,w} \le Cm\|v\|_{r,w},$$

$$\|b\otimes v\|_{\frac{3r}{3+r},w} \le C\|v\|_{r,w}\|b\|_{3,w} \le Cm\|v\|_{r,w}.$$

Therefore,

$$\|H(v)\|_{\frac{3r}{3+r},w} \le C\big(\|v\|_{3,w} + \|\hat{z}\|_{3,w} + 2m\big)\|v\|_{r,w}. \quad (33)$$

Then the first integral in (32) can be estimated as

$$\int_0^{\frac{t}{2}} |\langle -H(v)(t-\xi), \nabla U(t,t-\xi)^*\varphi\rangle| d\xi \le \int_0^{\frac{t}{2}} \|H(v)(t-\xi)\|_{\frac{3r}{3+r},w}\|\nabla U(t,t-\xi)^*\varphi(t)\|_{\frac{3r}{2r-3},1} d\xi$$
$$\le \int_0^{\frac{t}{2}} C\big(\|v(t-\xi)\|_{3,w} + \|\hat{z}(t-\xi)\|_{3,w} + 2m\big)\|v(t-\xi)\|_{r,w}.\|\nabla U(t,t-\xi)^*\varphi(t)\|_{\frac{3r}{2r-3},1} d\xi$$
$$\le C\big(\|v\|_{\infty,3,w} + \|\hat{z}\|_{\infty,3,w} + 2m\big)\int_0^{\frac{t}{2}} (t-\xi)^{-\frac{1}{2}+\frac{3}{2r}}(t-\xi)^{\frac{1}{2}-\frac{3}{2r}}\|v(t-\xi)\|_{r,w}\|\nabla U(t,t-\xi)^*\varphi(t)\|_{\frac{3r}{2r-3},1} d\xi$$
$$\le C\big(\|v\|_{\mathbb{M}} + \|\hat{z}\|_{\infty,3,w} + 2m\big)\|v\|_{\mathbb{M}}\int_0^{\frac{t}{2}} (t-\xi)^{-\frac{1}{2}+\frac{3}{2r}}\|\nabla U(t,t-\xi)^*\varphi(t)\|_{\frac{3r}{2r-3},1} d\xi$$
$$\le C\left(\frac{t}{2}\right)^{-\frac{1}{2}+\frac{3}{2r}}\big(\|v\|_{\mathbb{M}} + \|\hat{z}\|_{\infty,3,w} + 2m\big)\|v\|_{\mathbb{M}}\int_0^{\frac{t}{2}} \|\nabla U(t,t-\xi)^*\varphi(t)\|_{\frac{3r}{2r-3},1} d\xi.$$

We use estimate (9) to obtain

$$\int_0^{\frac{t}{2}} \|\nabla U(t,t-\xi)^*\varphi(t)\|_{\frac{3r}{2r-3},1} d\xi \le C\|\varphi(t)\|_{\frac{r}{r-1},1}.,$$

Thus,

$$\int_0^{\frac{t}{2}} |\langle -H(v)(t-\xi), \nabla U(t,t-\xi)^*\varphi\rangle| d\xi \le C\left(\frac{t}{2}\right)^{-\frac{1}{2}+\frac{3}{2r}}\big(\|v\|_{\mathbb{M}} + \|\hat{z}\|_{\infty,3,w} + 2m\big)\|v\|_{\mathbb{M}}\|\varphi(t)\|_{\frac{r}{r-1},1} \quad (34)$$

Similarly (33) we have

$$\|H(v)(t-\xi)\|_{\frac{3}{2},w} \le C\big(\|v(t-\xi)\|_{3,w} + \|\hat{z}(t-\xi)\|_{3,w} + 2m\big)\|v(t-\xi)\|_{3,w} \quad (35)$$

Then the second integral in (32) can be calculated as

$$\int_{t/2}^{t} |\langle -H(v)(t-\xi), \nabla U(t,t-\xi)^* \varphi \rangle| d\xi \le \int_{\frac{t}{2}}^{t} \|H(v)(t-\xi)\|_{\frac{3}{2},w} \|\nabla U(t,t-\xi)^* \varphi(t)\|_{3,1} d\xi$$

$$\le C \int_{\frac{t}{2}}^{t} \big(\|v(t-\xi)\|_{3,w} + \|\hat{z}(t-\xi)\|_{3,w} + 2m\big) \|v(t-\xi)\|_{3,w} \|\nabla U(t,t-\xi)^* \varphi(t)\|_{3,1} d\xi$$

$$\le C\big(\|v\|_{\mathbb{M}} + \|\hat{z}\|_{\infty,3,w} + 2m\big)\|v\|_{\mathbb{M}} \int_{\frac{t}{2}}^{t} \xi^{-\frac{3}{2}+\frac{3}{2r}} \|\varphi(t)\|_{\frac{r}{r-1},1} d\xi$$

$$\le C(t)^{-\frac{1}{2}+\frac{3}{2r}}\big(\|v\|_{\mathbb{M}} + \|\hat{z}\|_{\infty,3,w} + 2m\big)\|v\|_{\mathbb{M}}\|\varphi\|_{\frac{r}{r-1},1}. \quad (36)$$

Lastly, (32), (33), and (34) altogether yield

$$\left|\left\langle \int_0^t U(t,t-\xi)\, \mathbb{P}\mathrm{div}\big(H(v)(t-\xi)\big) d\xi, \varphi \right\rangle\right| \le \tilde{C}(t)^{-\frac{1}{2}+\frac{3}{2r}}\big(\|v\|_{\mathbb{M}} + \|\hat{z}\|_{\infty,3,w} + 2m\big)\|v\|_{\mathbb{M}}\|\varphi\|_{\frac{r}{r-1},1}. \quad (37)$$

For all $\varphi \in C_{0,\sigma}^{\infty}(\Omega)$. Therefore,

$$(t)^{\frac{1}{2}-\frac{3}{2r}} \left\| \int_0^t U(t,t-\xi)\, \mathbb{P}\mathrm{div}\big(H(v)(t-\xi)\big) d\xi \right\|_{r,w}$$

$$\le \tilde{C}\big(\|v\|_{\mathbb{M}} + \|\hat{z}\|_{\infty,3,w} + 2m\big)\|v\|_{\mathbb{M}} \quad (38)$$

For all $t > 0$ yielding that

$$\|\Phi(v)\|_{\mathbb{M}} = \left\| U(t,0)\big(u(0) - \hat{z}(0)\big) + \int_0^t U(t,\tau)\, \mathbb{P}\mathrm{div}\big(H(v)\big) d\tau \right\|_{\mathbb{M}}$$

$$\le \big\|U(t,0)\big(u(0)-\hat{z}(0)\big)\big\|_{\mathbb{M}} + \left\| \int_0^t U(t,\tau)\, \mathbb{P}\mathrm{div}\big(H(v)\big) d\tau \right\|_{\mathbb{M}}$$

$$\le C\|u(0) - \hat{z}(0)\|_{3,w} + \tilde{C}\big(\|v\|_{\mathbb{M}} + \|\hat{z}\|_{\infty,3,w} + 2m\big)\|v\|_{\mathbb{M}}. \quad (39)$$

In a same way as above, we arrive at

$$\|\Phi(v_1) - \Phi(v_1)\|_{\mathbb{M}} \le C\big(\|v_1\|_{\mathbb{M}} + \|v_2\|_{\mathbb{M}} + 2\|\hat{z}\|_{\infty,3,w} + 2m\big)\|v_1 - v_2\|_{\mathbb{M}}.$$

for $v_1, v_2 \in \mathbb{M}$.

Hence, for sufficiently small $\|u(0) - \hat{z}(0)\|_{3,w}$, $\|\hat{z}\|_{\infty,3,w}, m$ and ρ, the mapping Φ maps from $\mathcal{B}_\rho$ into $\mathcal{B}_\rho$, and it is a contraction. So, Φ has a unique fixed point. Therefore, the function $v = u - \hat{z}$, being the fixed-point of this mapping, belongs to $\mathbb{M}$. Thus, we obtain (25), and hence the stability of $\hat{z}$ follows.

4. Conclusion

This paper we study Navier- Stokes flow in the exterior of a moving and rotating obstacle. Particular emphasis is placed on the fact that the motion of the obstacle is non-autonomous, i.e. the translational and angular velocities depend on time. Then a change of variables yields a new modified non-autonomous Navier-Stokes systems of Oseen type if the velocity at infinity is nonzero - with nontrivial perturbation terms.

Our techniques use known $L^p - L^q$ estimates of the evolution family and its gradient for the linear parts and fixed-point arguments. We prove boundedness and polynomial stability of mild solutions when the initial data belong to L^p_σ and are sufficiently small.

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